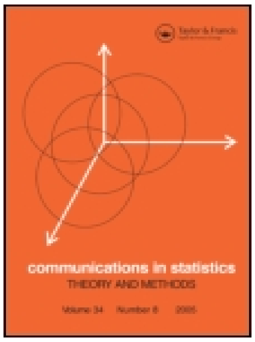

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Test of Normality Against Generalized Exponential Power Alternatives

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The family of symmetric generalized exponential power (GEP) densities offers a wide range of tail behaviors, which may be exponential, polynomial, and/or logarithmic. In this article, a test of normality based on Rao's score statistic and this family of GEP alternatives is proposed. This test is tailored to detect departures from normality in the tails of the distribution. The main interest of this approach is that it provides a test with a large family of symmetric alternatives having non-normal tails. In addition, the test's statistic consists of a combination of three quantities that can be interpreted as new measures of tail thickness. In a Monte-Carlo simulation study, the proposed test is shown to perform well in terms of power when compared to its competitors.

Keywords Generalized exponential power; Rao's score test; Symmetric distributions; Tail behavior; Test of normality.

Mathematics Subject Classification 62E20; 62E25; 62F03; 60F05.

1. Introduction

An important issue in applied statistics is the validity of the normality assumptions that are often required for the use of many popular methods of statistical analysis. Consequently, the problem of testing that a sample has been drawn from some normal distribution with unknown mean and variance is one of the most common problems of goodness of fit in statistical practice. For this reason, many test procedures have been proposed in the literature. These tests can be mainly divided into three groups.

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In the first group, the observed sample distribution is compared to the normal distribution using some measure of discrepancy. This can be done in different ways, one of which is the computation of a distance between the empirical distribution function (edf) and the normal cumulative distribution function (cdf). To accomplish this task, different norms have been used. The tests proposed by Cramér (1928), Kolmogorov (1933), and Anderson and Darling (1954) are very famous examples of this; see also Zhang and Wu (2005). Another approach consists of assessing the closeness between a nonparametric estimate of the density (usually a histogram, or some kernel estimator) and the normal density function. Again, different norms have been used for this. Pearson's χ^2 test could be put into this category; see also Fan (1994) and Cao and Lugosi (2005). Other approaches have also been considered. For instance, we would also include in this group the procedure credited to Epps and Pulley (1983) which is based on the empirical characteristic function.

Moment-based tests form the second group. Tests based on skewness or Pearson's kurtosis, or a combination of both, naturally fall into this group; see, for example, the tests proposed by D'Agostino and Pearson (1973), Bowman and Shenton (1975), Jarque and Bera (1987), and D'Agostino et al. (1990). We would also include here the test based on the ratio of the mean (absolute) deviation to the standard deviation due to Geary (1935) and the test based on G-Kurtosis introduced by Bonett and Seier (2002).

The third group is composed of regression and correlation tests; see, for example, Shapiro and Wilk (1965), D'Agostino (1971), Filliben (1975), Gan and Koehler (1990), and Coin (2008). The Shapiro-Wilk test is quite possibly the most well-known and popular test of normality. It has also been recognized as the best available omnibus test by different authors who compared the power of different tests for many different alternatives; see Shapiro et al. (1968), Pearson et al. (1977), and D'Agostino and Stephens (1986, Sec. 9.4). Many authors have worked on extensions of the Shapiro-Wilk test, including Shapiro and Francia (1972), Royston (1989, 1992), and Rahman and Govindarajulu (1997). Most of the tests in this group use a test statistic that can essentially be viewed as the squared correlation R^2 obtained from a normal probability plot, that is, from a regression of the inverse normal cdf of the edf on the observations. The key here, is that R^2 is expected to be close to unity when the sample comes from a normal distribution.

There are, however, many other tests of normality in the literature, all based on different ideas and characterizations of the normal distribution. Among them, we mention the test of the ratio of the range to the standard deviation due to David et al. (1954) and the tests related to entropy and the Kullback-Leibler information due to Vasicek (1976), Arizono and Ohta (1989), and Park (1999); see also Locke and Spurrier (1976), Spiegelhalter (1977), Oja (1983), Lariccia (1986), Mudholkar et al. (2002), and Gel et al. (2007). For a comprehensive review of goodness-of-fit tests in general and in the particular case of normality, we refer the reader to D'Agostino and Stephens (1986) and Thode (2002). Finally, more recent articles on testing for normality are Carota (2010), Drezner et al. (2010), Goia et al. (2011), Han (2010), Noughabi (2010), Sadooghi-Alvandi and Rasekhi (2009), and Zghoul (2010); see also the comprehensive simulation studies by Thadewald and Büning (2007) and Romão et al. (2010).

In this article, a new test of normality is proposed. This test is designed to detect non normal tail behavior by considering a large family of symmetric alternatives and hence, is directional in nature. The strategy we adopt here, similar to the one used by Neyman (1937) or later by Mardia and Kent (1991), consists of using Rao's score test (also known as the Lagrange multiplier test, cf. Rao [1948]) and the generalized exponential power (GEP) family of densities which can exhibit a large range of tail behaviours (cf. Desgagné and Angers, 2005). Specifically, we develop a test of normality with unknown location and scale based on Rao's score by taking this vast family of densities as an alternative. Note that in the simple case where location and scale are considered known, our procedure is equivalent to a test of a simple null hypothesis about the GEP parameters. The resulting test statistic can be seen as a combination of three measures of tail behavior coming together to form an original and simple test statistic having an asymptotically χ^2 distribution with three degrees of freedom under the null hypothesis.

For the remainder of this article, we proceed as follows. In Sec. 2, the family of GEP densities is presented and adapted for the purpose of our test. In Sec. 3, Rao's score test on the family of modified GEP densities is used to obtain a test of normality with unkown location and scale. In Sec. 4, a Monte Carlo simulation study is conducted to compare the power of the proposed test with the power of other known tests of normality over a wide range of alternatives. Finally, we briefly conclude in Sec. 5.

2. Generalized Exponential Power Densities

The family of generalized exponential power densities covers a large range of tail behaviors, ranging from exponential to polynomial and logarithmic. This family of distributions has been considered in applications where tail behavior is of special interest, such as Bayesian robustness (see Desgagné and Angers, 2007), the characterization of tails of distributions using p-credence (see Angers, 2000), and the selection of an importance function in Monte Carlo simulations (see Desgagné and Angers, 2005). In Sec. 3, we will use this family of densities as an embedding family of alternatives to develop a test of normality. This strategy can be adopted because the normal density is included in the family of GEP densities. This idea is not new and dates back to Neyman (1937). The main interest in the present approach is to provide a simple and powerful test against symmetric alternatives.

Desgagné and Angers (2005) defined the GEP density as having the following general form:

$$p(x; \gamma, \delta, \alpha, \beta, z_0) \propto e^{-\delta|x|^\gamma}|x|^{-\alpha}(\log|x|)^{-\beta} \quad \text{if } |x| \geq z_0,$$

and equal to $p(z_0; \gamma, \delta, \alpha, \beta, z_0)$ for $|x| < z_0$. Notice that the constant $p(z_0; \gamma, \delta, \alpha, \beta, z_0)$ is selected to make the density continuous. The density is symmetric with respect to the origin and defined on the real line. The parameter space is a subset of $\mathbb{R}^5$ with other specific conditions on the parameters guaranteeing that the density is proper, positive, bounded, and unimodal. For more details, see Desgagné and Angers (2005).

The exponential term, which includes the parameters γ and δ, provides a large spectrum of tail behaviors. The polynomial term, controlled by the parameter α, provides densities with heavy tails. The logarithmic term, using the parameter β, provides densities with even heavier tails, sometimes called super heavy tails. The parameter z_0 has little impact on the tails of the density.

Certain known distributions are special cases of the GEP density. If the parameters α, β, and z_0 are set to 0, we get the exponential power density introduced by Box and Tiao (1962). In addition, if the parameter γ is set to 2, the normal density is obtained, and if it is instead set to 1, we get the Laplace density. Also, the right tail of the GEP density (for $x \geq z_0$) corresponds to certain known distributions. For example, the Weibull density is obtained by letting $\alpha = 1 - \gamma$ and $\beta = z_0 = 0$. Setting $\gamma = 1$, $\beta = z_0 = 0$ and imposing the constraint $\alpha < 1$ gives the gamma density. The generalized gamma, Rayleigh, and Maxwell-Boltzmann densities can be obtained similarly. If the exponential term is removed (when $\gamma = \delta = 0$), the GEP density is considered as a heavy-tailed distribution, with polynomial and possibly logarithmic terms. For example, if we set $\gamma = \delta = \beta = 0$ and impose the constraint $\alpha > 1$, the Pareto density is obtained. If we instead set $\gamma = \delta = 0$ and $z_0 = 1$ and impose the constraints $\alpha > 1$ and $\beta < 1$, we get the log-gamma density, an exponential transformation of the gamma distribution. Letting $\gamma = \delta = 0$ and $\alpha = 1$ and imposing the constraints $\beta > 1$ and $z_0 > 1$ leads to the super heavy-tailed log-Pareto density, an exponential transformation of the Pareto distribution. Finally, note that GEP densities can approximate many classical densities quite well in the tails, suggesting that a test of normality against GEP alternatives should fare well against many non-GEP alternatives.

We now introduce a modified version of the GEP density to be used as a family of alternatives for our test. This is done by setting certain parameters to fixed values and by adding carefully chosen constants. Our rationale for doing this is the following.

First, note that the normal density $N(0, \sigma^2)$ is a special case of the GEP density, as it is defined above, when $\gamma = 2$, $\delta = \frac{1}{2\sigma^2}$, $\alpha = 0$, $\beta = 0$, $z_0 = 0$. This means that δ is a function of the scale parameter when normality is assumed. This is not desirable as the alternative family will be extended to a location-scale family in the next section, our test statistic incorporating classical maximum likelihood estimates of the location and scale parameters under normality. In this context, we fix δ to avoid an additional degenerate term when considering the score statistic (see (3) in the next section) and its use for the composite case. We chose to set $\delta = \frac{1}{2}$ to include the $N(0, 1)$ as a special case.

Secondly, it can be seen that the impact of the parameter z_0 is greatest around the origin since it essentially determines the constant part of the density. Its impact on the tail behavior, and consequently, its usefulness for detecting light and/or heavy tails can be expected to be limited. Thus, here we set $z_0 = 0$.

Finally, to ensure that the density remains positive and bounded for any values of the parameters γ, α and β, the constants 1 and e are added, respectively, to the polynomial and logarithmic terms. Note that these constants have limited impact on the tail behavior of the densities.

The modified family of densities that results can be written in the following way. We will write $X \sim \mathrm{GEP}(\boldsymbol{\theta})$ when the density of X is given by

$$g(x; \boldsymbol{\theta}) = k(\boldsymbol{\theta}) \mathrm{e}^{-\frac{1}{2}|x|^{\theta_1}} \big(1 + |x|\big)^{-\theta_2} \big(\log(\mathrm{e} + |x|)\big)^{-\theta_3}, \tag{1}$$

for all $x \in \mathbb{R}$, where $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)^T$ is the vector of parameters, $k(\boldsymbol{\theta})$ is the normalizing constant, and $\theta_1 \in \mathbb{R}^+$, $\theta_2 \in \mathbb{R}$, and $\theta_3 \in \mathbb{R}$. Note that the normalizing constant is generally not available in closed form, although it is in simple cases. Also, the conditions $\theta_2 \geq 1$ if $\theta_1 = 0$, and $\theta_3 > 1$ if both $\theta_1 = 0$ and $\theta_2 = 1$, must be satisfied for the density to be proper. We denote the resulting parameter space by $\boldsymbol{\Theta}$. Finally, note that the standard normal density corresponds to the special case where $\boldsymbol{\theta} = (2, 0, 0)^T$, and that this point is located in the interior of the parameter space.

3. The Proposed Test for Normality

Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with density

$$f(x; \boldsymbol{\theta}, \mu, \sigma) = \frac{1}{\sigma} g\big((x-\mu)/\sigma; \boldsymbol{\theta}\big),$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$ are, respectively, the unknown location and scale parameters, and $g(x; \boldsymbol{\theta})$ is defined as in (1). We will write $X \sim \mathrm{GEP}(\boldsymbol{\theta}; \mu, \sigma)$ when the density of X is given by $f(x; \boldsymbol{\theta}, \mu, \sigma)$.

Our goal here is to carry out a test that the measurements of X come from some $N(\mu, \sigma^2)$ distribution (with μ and σ unspecified), considering GEP distributions as a family of alternatives or, equivalently, assuming that these measurements come from some $\mathrm{GEP}(\boldsymbol{\theta}; \mu, \sigma)$ distribution. In other words, we want to test

$$\begin{aligned} H_0 &: \mathscr{L}(X) \in \{N(\mu, \sigma^2); \mu \in \mathbb{R}, \sigma > 0\} = \{\mathrm{GEP}(2, 0, 0; \mu, \sigma); \mu \in \mathbb{R}, \sigma > 0\}, \\ &\quad \text{against} \\ H_1 &: \mathscr{L}(X) \in \{\mathrm{GEP}(\boldsymbol{\theta}; \mu, \sigma); \boldsymbol{\theta} \in \boldsymbol{\Theta}/\{(2, 0, 0)^T\}, \mu \in \mathbb{R}, \sigma > 0\}, \end{aligned} \tag{2}$$

with $\boldsymbol{\Theta}$ denoting the parameter space introduced in Sec. 2. Different approaches could be used to achieve this. Maximum likelihood techniques obviously come to mind, but the fact that the normalizing constant $k(\boldsymbol{\theta})$ in (1) is, in general, not analytically tractable makes the use of these techniques difficult. We here propose to take a different approach and make use of Rao's score test, considering μ and σ as nuisance parameters. If we define

$$\boldsymbol{d}_0(y) = \frac{\partial}{\partial \boldsymbol{\theta}} \log g(y; \boldsymbol{\theta})\Big|_{\boldsymbol{\theta}=(2,0,0)^T},$$

it can be shown that

$$\boldsymbol{d}_0(y) = \begin{pmatrix} 0.18240929 - \frac{1}{2} y^2 \log|y| \\ 0.53482230 - \log(1+|y|) \\ 0.20981558 - \log\big(\log(\mathrm{e}+|y|)\big) \end{pmatrix}. \tag{3}$$

Then the score vector to be used for our test is defined, for given values of μ and σ, as

$$\boldsymbol{r}_n(\mu, \sigma) = \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{d}_0(Y_i), \tag{4}$$

where $Y_i = (X_i - \mu)/\sigma$. Note that the three included constants ensure that each component of the score vector has zero mean under normality. Note also that the first element of the vector in (3) is not defined for $Y_i = 0$. However, by convention, we set it to zero when $Y_i = 0$ since $\lim_{y\to 0}(y^2 \log|y|) = 0$. It is worth noting that $\boldsymbol{r}_n(\mu, \sigma)$ consists of three new measures of tail thickness given by

$$\boldsymbol{r}_n^{(1)}(\mu, \sigma) = 0.18240929 - \frac{1}{2n}\sum_{i=1}^{n} Y_i^2 \log|Y_i|,$$

$$\boldsymbol{r}_n^{(2)}(\mu, \sigma) = 0.53482230 - \frac{1}{n}\sum_{i=1}^{n} \log(1 + |Y_i|),$$

$$\boldsymbol{r}_n^{(3)}(\mu, \sigma) = 0.20981558 - \frac{1}{n}\sum_{i=1}^{n} \log\big(\log(\mathrm{e} + |Y_i|)\big).$$

As pointed out by a referee, it would be interesting to study more specifically the behavior of the above quantities and compare them, for example, to Pearson's sample kurtosis or to G-Kurtosis (cf. Bonett and Seier, 2002) in how they assess thickness of the tails of specific distributions.

Now, under the null hypothesis of normality, the results of Rao (1948) imply that

$$n^{1/2}\boldsymbol{r}_n(\mu, \sigma) \xrightarrow{\mathcal{D}} N_3(\boldsymbol{0}, \boldsymbol{J}_0) \quad \text{and} \quad n\boldsymbol{r}_n(\mu, \sigma)^T \boldsymbol{J}_0^{-1} \boldsymbol{r}_n(\mu, \sigma) \xrightarrow{\mathcal{D}} \chi_3^2, \tag{5}$$

where the information matrix $\boldsymbol{J}_0$ is defined as

$$\boldsymbol{J}_0 = \mathbb{E}_0\big[\boldsymbol{d}_0(Y)\boldsymbol{d}_0(Y)^T\big],$$

where $Y = (X - \mu)/\sigma$ and the notation $\mathbb{E}_0[\cdot]$ indicates that the expectation is taken under the null hypothesis. It is not difficult to show that

$$\boldsymbol{J}_0 = \begin{pmatrix} 0.42423099 & 0.13285410 & 0.05401580 \\ 0.13285410 & 0.10080860 & 0.04056528 \\ 0.05401580 & 0.04056528 & 0.01632673 \end{pmatrix}, \tag{6}$$

doing the involved numerical calculations by using Monte-Carlo simulations or any appropriate numerical integration method.

However, note that the result in (5) has no practical utility since, μ and σ being unknown, $\boldsymbol{r}_n(\mu, \sigma)$ cannot be computed from the observations at hand. To circumvent this, the strategy we adopt is quite natural, and similar to the one adopted by Jarque and Bera (1987). Essentially, we still rely on the score vector, but use the maximum likelihood estimators of μ and σ under normality, respectively,

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad S_n = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2},$$

as surrogates for the unknown μ and σ. Hence, we simply propose to base the composite test of normality on $\boldsymbol{r}_n(\overline{X}_n, S_n)$. However, the repercussions of this substitution have to be well understood. The remainder of this section is devoted to obtaining the asymptotic behavior of $\boldsymbol{r}_n(\overline{X}_n, S_n)$ and of the resulting test statistic.

For this, we first establish the (asymptotic) link between $\boldsymbol{r}_n(\overline{X}_n, S_n)$ and $\boldsymbol{r}_n(\mu, \sigma)$ so as to make use of the results given in (5) above. An advantage of using this approach is that it explicitly describes the effect of using $\overline{X}_n$ and S_n in lieu of the unknown parameters when computing the test statistic.

Proposition 3.1. *When $X \sim N(\mu, \sigma^2)$, we have that*

$$\boldsymbol{r}_n(\overline{X}_n, S_n) = \boldsymbol{r}_n(\mu, \sigma) - \frac{1}{2}(1 - T_n)\boldsymbol{v}_0 + o_P\left(n^{-\frac{1}{2}}\right)\mathbf{1}_3,$$

where

$$T_n = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i - \mu}{\sigma}\right)^2,$$

and where $\mathbf{1}_3 = (1, 1, 1)^T$ and

$$\boldsymbol{v}_0 = -\mathbb{E}_0\left[Y^2\boldsymbol{d}_0(Y)\right] = (0.86481850, 0.38512925, 0.15594892)^T. \tag{7}$$

A proof of this result is given in the Appendix with the vector $\boldsymbol{v}_0$ requiring the use of numerical integration to be calculated.

Based on the previous result, it is now possible to determine the asymptotic distribution of $\boldsymbol{r}_n(\overline{X}_n, S_n)$ and to present the statistic to be used for the composite test of normality, denoted hereafter by R_n. The presentation of a proof of Theorem 3.1 is delayed to the Appendix. This is our main result.

Theorem 3.1. *The modified score vector for the composite test of normality of the hypotheses given in* (2) *is $\boldsymbol{r}_n(\overline{X}_n, S_n)$ where $\boldsymbol{r}_n$ is defined as in* (4). *Furthermore, under the null hypothesis,*

$$n^{1/2}\boldsymbol{r}_n(\overline{X}_n, S_n) \xrightarrow{\mathcal{D}} N_3\left(\mathbf{0}, \boldsymbol{J}_0 - \frac{1}{2}\boldsymbol{v}_0\boldsymbol{v}_0^T\right),$$

where $\boldsymbol{J}_0$ is given in (6) *and the vector $\boldsymbol{v}_0$ is given in* (7). *The proposed statistic to test the composite hypothesis of normality is given by*

$$R_n = n\boldsymbol{r}_n(\overline{X}_n, S_n)^T\left(\boldsymbol{J}_0 - \frac{1}{2}\boldsymbol{v}_0\boldsymbol{v}_0^T\right)^{-1}\boldsymbol{r}_n(\overline{X}_n, S_n),$$

and, under the null hypothesis, we have that $R_n \xrightarrow{\mathcal{D}} \chi^2_3$.

Note that the test statistic is location and scale invariant. Note also that the asymptotic covariance matrix of $n^{1/2}\boldsymbol{r}_n(\overline{X}_n, S_n)$ is given by

$$\boldsymbol{J}_0 - \frac{1}{2}\boldsymbol{v}_0\boldsymbol{v}_0^T = \begin{pmatrix} 0.0502754623 & -0.0336793487 & -0.0134179540 \\ -0.0336793487 & 0.0266463308 & 0.0105350321 \\ -0.0134179540 & 0.0105350321 & 0.00416669944 \end{pmatrix},$$

which is quite different from the asymptotic covariance matrix of $n^{1/2}\boldsymbol{r}_n(\mu, \sigma)$ given in (6).

We next study the power of the resulting procedure through some simulations. Comparisons are made with many other tests that are available in the literature.

4. Practical Considerations and Simulations

In this section, we first give a formula to compute quantiles and p-values for our test for finite sample sizes. This is done since the convergence in distribution of the statistic R_n seems very slow. Then, we compare the power of the proposed test for the composite hypothesis of normality with selected tests for different symmetric alternatives.

In order to see if the distribution of R_n, for finite samples, can be approximated by the Chi-square distribution with three degrees of freedom, we estimated the power of our test under the null hypothesis of normality, using the Chi-square quantiles, and we observed that the results were too far from the exact level, for different levels and sample sizes. We concluded that the use of simulated quantiles, or of a formula that estimates these quantiles, would be necessary in practice. To derive such a formula, we simulated quantiles of different orders for different sample sizes and were able to find a simple formula to estimate all these quantiles. Specifically, we proceeded as follows.

For each sample size n given in Table 1, we found the best regression of the quantiles of order α explained by a power of α, ranging from 0.01–0.50. That is, we

Table 1
Set of parameters a_n, b_n, c_n for the two formulas (8) and (9), valid for $0.01 \leq p$-value, $\alpha \leq 0.50$

n	a_n	b_n	c_n	n	a_n	b_n	c_n
10	−90.771	91.196	−0.02	270	−15.022	15.936	−0.12
20	−21.640	22.203	−0.08	280	−14.979	15.906	−0.12
30	−12.320	13.022	−0.13	290	−16.889	17.774	−0.11
40	−10.463	11.218	−0.15	300	−16.848	17.744	−0.11
50	−8.805	9.650	−0.17	310	−16.820	17.725	−0.11
60	−8.957	9.807	−0.17	320	−16.778	17.692	−0.11
70	−9.072	9.928	−0.17	330	−19.069	19.940	−0.10
80	−8.279	9.200	−0.18	340	−19.028	19.908	−0.10
90	−9.154	10.043	−0.17	350	−18.998	19.886	−0.10
100	−9.163	10.071	−0.17	360	−18.970	19.864	−0.10
110	−9.153	10.082	−0.17	370	−18.936	19.838	−0.10
120	−10.101	10.999	−0.16	380	−18.893	19.805	−0.10
130	−10.083	10.999	−0.16	390	−21.672	22.539	−0.09
140	−10.059	10.993	−0.16	400	−21.639	22.513	−0.09
150	−11.128	12.028	−0.15	410	−21.609	22.489	−0.09
160	−11.098	12.015	−0.15	420	−21.580	22.465	−0.09
170	−11.070	12.002	−0.15	430	−21.548	22.440	−0.09
180	−11.031	11.980	−0.15	440	−25.013	25.860	−0.08
190	−12.240	13.153	−0.14	450	−24.976	25.829	−0.08
200	−12.198	13.127	−0.14	460	−24.956	25.813	−0.08
210	−12.160	13.104	−0.14	470	−24.928	25.790	−0.08
220	−13.551	14.457	−0.13	480	−24.897	25.765	−0.08
230	−13.507	14.428	−0.13	490	−24.874	25.745	−0.08
240	−13.468	14.402	−0.13	500	−24.848	25.724	−0.08
250	−15.096	15.988	−0.12	1000	−54.342	55.166	−0.04
260	−15.058	15.963	−0.12	∞	64.056	−63.392	0.04

found the values of a_n, b_n, and c_n that maximize the R^2 of the regression

$$q_{\alpha;n} = a_n + b_n \alpha^{c_n}, \tag{8}$$

where $q_{\alpha;n}$ is the simulated quantile such that $\Pr[R_n > q_{\alpha;n}] \approx \alpha$. Doing this, we found the fit to be remarkably good, R^2 values ranging from 0.9995—0.9999 for all the considered sample sizes. The coefficients a_n and b_n and the exponent c_n are given in Table 1. For example, if $n = 50$ and the chosen level of the test is $\alpha = 0.05$, then the hypothesis of normality of the observations should be rejected if

$$R_{50} > q_{0.05;50} = -8.805 + 9.650(0.05)^{-0.17} = 7.2534.$$

Furthermore, a formula for p-values between 0.01 and 0.50 can easily be obtained by inverting (8). This leads to

$$p\text{-value} \approx \left(\frac{R_n - a_n}{b_n}\right)^{1/c_n}, \tag{9}$$

where R_n is the observed value of the test statistic. For example, if we observed $R_{50} = 6.02$, the approximate p-value of the test would be

$$p\text{-value} \approx \left(\frac{6.02 - (-8.805)}{9.650}\right)^{1/(-0.17)} = 0.08,$$

and the hypothesis of normality would not be rejected at a level of $\alpha = 0.05$. In addition to the excellent fit of the regressions to estimate the quantiles, we assessed the quality of the fit by performing a study of the empirical power under the null of R_n based on 1,000,000 simulations and on quantiles calculated from (8) and from the coefficients given in Table 1. The results given in Table 2 suggest that the level of the test is very accurate using the formula.

Next, we performed simulations to compare the power of our proposed test based on R_n for the composite hypothesis of normality with selected tests for different symmetric alternatives. We performed an extensive empirical power study of tests for normality using all the test-competitors considered in Romão et al. (2010) in addition to our test. This study can be found in Desgagné et al. (2011). For this article, we selected only a sample of the most performant and known tests to keep it parsimonious, the conclusions remaining essentially the same.

These tests are the Anderson-Darling ($AD*$), Zhang-Wu Z_C test, D'Agostino-Pearson K^2 test, Jarque-Bera test (JB), Bonett-Seier test (T_w), Cabaña-Cabaña test ($T_{K,5}$), Shapiro-Wilk test (W), Chen-Shapiro test (CS), del Barrio-Cuesta-Albertos-Matrán-Rodríguez-Rodríguez quantile correlation test ($BCMR$), β_3^2 Coin test, and Gel-Miao-Gastwirth test (R_{sJ}). We refer the reader to Romão et al. (2010) for details and references on these tests. We also added the test of Spiegelhalter (1977) (S), which is also a directional test for symmetric alternatives.

Different symmetric alternatives have been chosen. We first studied 142 GEP alternatives to cover a large spectrum of tail behaviours. Again this study can be found in Desgagné et al. (2011). For this article, we selected a smaller but still representative sample of 73 GEP alternatives, the conclusions remaining essentially the same. They are listed in Tables 3 and 4. For each of the following kurtosis β_2

Table 2

Empirical power under the null of R_n based on 100,000 simulations and on quantiles given by formula in Table 1

n and $\alpha \times 100$	1	2	3	4	5	6	7	8	9	10	25	30	35	40	45	50
10	1.03	2.00	3.00	4.01	5.00	5.94	6.85	7.84	9.06	9.83	24.84	30.15	35.14	40.13	45.03	49.78
20	1.03	2.03	2.97	4.09	5.02	5.96	7.07	8.04	9.09	10.11	25.29	30.51	35.21	39.85	44.56	49.04
30	0.98	1.88	2.88	3.91	4.86	5.87	6.98	8.01	9.06	10.17	25.59	30.57	35.37	39.98	44.40	48.76
40	1.00	1.98	2.91	3.93	4.99	6.05	7.11	7.98	9.27	10.35	25.71	30.69	35.38	39.77	44.43	48.77
50	1.04	1.98	2.95	3.88	5.05	5.86	7.11	8.24	9.33	10.33	25.57	30.45	34.92	39.39	43.87	48.08
60	1.06	1.90	2.89	3.87	4.84	5.95	6.97	8.08	9.14	10.20	25.81	30.63	35.22	39.64	43.94	48.62
70	0.98	1.98	2.80	3.90	4.89	5.92	6.96	8.02	9.10	10.19	25.66	30.49	35.31	39.88	44.35	48.51
80	1.04	1.92	2.95	3.86	4.92	5.95	7.09	8.18	9.29	10.30	25.69	30.54	35.07	39.23	43.56	47.60
90	1.03	1.90	2.76	4.00	4.95	6.05	7.18	8.05	9.11	10.18	25.51	30.49	35.41	39.80	44.03	48.34
100	1.01	1.92	2.95	3.91	4.89	6.05	6.99	8.22	9.15	10.36	25.62	30.56	35.38	39.78	44.14	48.31
110	0.98	1.94	2.89	3.80	4.98	6.05	7.15	8.18	9.23	10.27	25.74	30.20	35.42	39.67	43.91	47.95
120	1.00	1.97	2.88	3.87	5.00	6.07	7.01	7.97	9.08	10.34	25.72	30.57	35.47	40.08	44.26	48.62
130	1.03	1.95	2.96	3.88	4.90	5.96	7.07	8.07	9.20	10.15	25.79	30.52	35.17	39.75	44.09	48.24
140	1.06	1.93	2.87	3.86	5.01	6.10	7.14	7.93	9.02	10.10	25.65	30.44	35.24	39.60	43.99	48.19
150	1.08	1.90	2.89	3.83	4.82	5.95	6.88	8.24	9.23	10.02	25.67	30.51	35.44	39.85	44.51	48.56
160	1.01	1.93	2.92	3.89	4.96	5.98	7.05	8.08	9.15	10.15	25.75	30.68	35.30	39.68	44.52	48.52
170	1.00	1.90	2.97	3.93	5.12	6.08	6.92	8.02	9.05	10.14	25.72	30.58	35.33	39.63	44.20	48.38
180	1.00	1.93	2.93	3.96	5.04	5.99	7.12	8.24	9.30	10.35	25.85	30.30	35.42	39.92	44.11	48.33
190	1.00	1.98	2.92	3.83	4.96	5.82	6.94	7.85	9.01	10.36	25.56	30.58	35.15	39.92	44.31	48.51
200	1.00	1.89	2.86	3.91	4.97	6.09	6.95	8.23	9.25	10.27	25.58	30.63	35.24	39.77	44.22	48.26
210	1.01	1.90	2.97	3.99	4.90	5.97	7.04	8.11	9.15	10.16	25.60	30.51	35.25	39.86	44.01	48.43
220	1.06	1.89	2.93	3.91	5.02	5.90	6.90	8.09	9.17	10.18	25.56	30.64	35.37	40.12	44.25	48.50
230	1.05	1.91	2.91	4.02	4.95	5.87	7.04	8.09	9.10	10.08	25.56	30.37	35.15	39.83	44.24	48.66
240	1.06	1.91	3.04	3.94	4.94	6.06	7.08	8.08	9.12	10.39	25.88	30.46	35.19	39.98	44.01	48.26
250	1.03	1.93	2.92	3.90	4.92	5.96	7.00	8.12	9.19	10.17	25.73	30.72	35.50	39.99	44.57	48.76

(*continued*)

Table 2
Continued

n and $\alpha \times 100$	1	2	3	4	5	6	7	8	9	10	25	30	35	40	45	50
260	0.96	1.88	2.96	3.82	4.97	5.92	6.96	7.99	9.01	10.16	25.62	30.59	35.29	39.94	44.28	48.65
270	1.02	1.93	2.78	3.90	4.95	6.05	7.09	8.12	9.14	10.23	25.54	30.53	35.45	39.90	44.27	48.38
280	1.01	1.98	2.99	3.93	4.96	6.02	7.07	8.12	9.09	10.14	25.58	30.07	35.34	39.89	44.31	48.61
290	1.01	1.97	2.98	4.00	4.93	5.99	6.93	7.96	9.09	10.21	25.66	30.53	35.34	39.94	44.63	48.78
300	1.03	1.97	2.96	3.96	4.89	6.01	6.94	8.14	9.05	10.23	25.68	30.68	35.41	39.75	44.34	48.56
310	1.00	1.99	2.99	4.05	4.97	5.98	7.03	8.10	9.19	10.24	25.79	30.29	35.39	40.01	44.21	48.62
320	0.98	1.96	2.96	3.98	4.99	5.97	7.06	8.08	9.20	10.31	25.61	30.59	35.25	39.75	44.18	48.64
330	1.01	2.00	3.02	3.92	4.89	6.02	7.07	8.13	9.00	10.17	25.71	30.54	35.45	40.10	44.70	48.98
340	0.98	1.95	2.95	4.06	4.97	5.99	7.01	8.02	8.92	10.16	25.68	30.57	35.27	40.02	44.28	48.78
350	0.96	1.97	2.88	3.96	4.89	5.93	6.95	8.05	9.16	10.17	25.86	30.43	35.40	39.87	44.36	48.60
360	1.04	1.95	2.93	4.02	4.95	6.01	6.92	8.08	9.17	10.22	25.45	30.46	35.31	39.96	44.11	48.61
370	1.00	1.91	2.90	3.91	4.98	5.96	7.04	8.02	9.08	10.14	25.39	30.37	35.12	39.83	44.00	48.61
380	1.07	1.89	3.03	3.81	5.18	5.92	7.02	8.00	9.08	10.13	25.51	30.35	35.26	39.78	44.16	48.44
390	1.04	1.98	2.92	3.94	4.83	5.91	6.87	7.97	8.98	10.03	25.70	30.65	35.57	40.17	44.64	48.52
400	0.96	1.90	2.91	3.91	4.84	5.93	6.92	7.98	9.05	10.21	25.48	30.43	35.21	39.80	44.51	48.67
410	0.96	1.93	2.91	3.89	4.92	5.86	6.94	7.94	9.05	10.22	25.58	30.40	35.27	39.93	44.45	48.56
420	0.95	1.90	2.92	3.97	4.87	5.93	6.96	7.95	9.16	10.00	25.39	30.40	35.17	39.74	44.16	48.41
430	0.95	1.88	2.97	3.84	4.86	6.06	6.95	7.97	8.99	10.22	25.61	30.41	35.29	39.86	44.25	48.48
440	0.97	1.99	2.86	3.86	4.81	5.84	6.94	7.98	9.05	10.07	25.63	30.59	35.50	40.38	44.79	48.96
450	1.00	1.91	2.85	3.89	4.85	6.04	7.03	8.08	9.08	10.11	25.56	30.22	35.28	39.99	44.42	48.78
460	1.11	1.86	2.87	3.88	5.01	6.04	6.97	8.02	9.02	10.15	25.66	30.57	35.34	39.95	44.50	48.75
470	0.99	1.96	2.95	4.00	4.92	5.77	6.97	8.07	9.25	10.18	25.54	30.40	35.23	39.80	44.25	48.66
480	1.05	1.95	2.96	3.91	5.15	5.90	7.02	7.99	9.02	10.18	25.86	30.42	35.34	39.60	44.59	48.66
490	0.95	1.90	2.96	3.92	4.90	5.92	6.92	8.22	8.95	10.01	25.42	30.37	35.10	40.15	44.21	48.84
500	0.95	1.91	2.91	3.94	5.07	5.94	6.91	7.92	8.95	10.01	25.86	30.81	35.31	39.82	44.35	48.67
1000	0.94	1.92	2.86	3.87	4.85	6.11	6.96	8.05	9.10	10.04	25.44	30.23	35.13	39.77	44.59	48.60

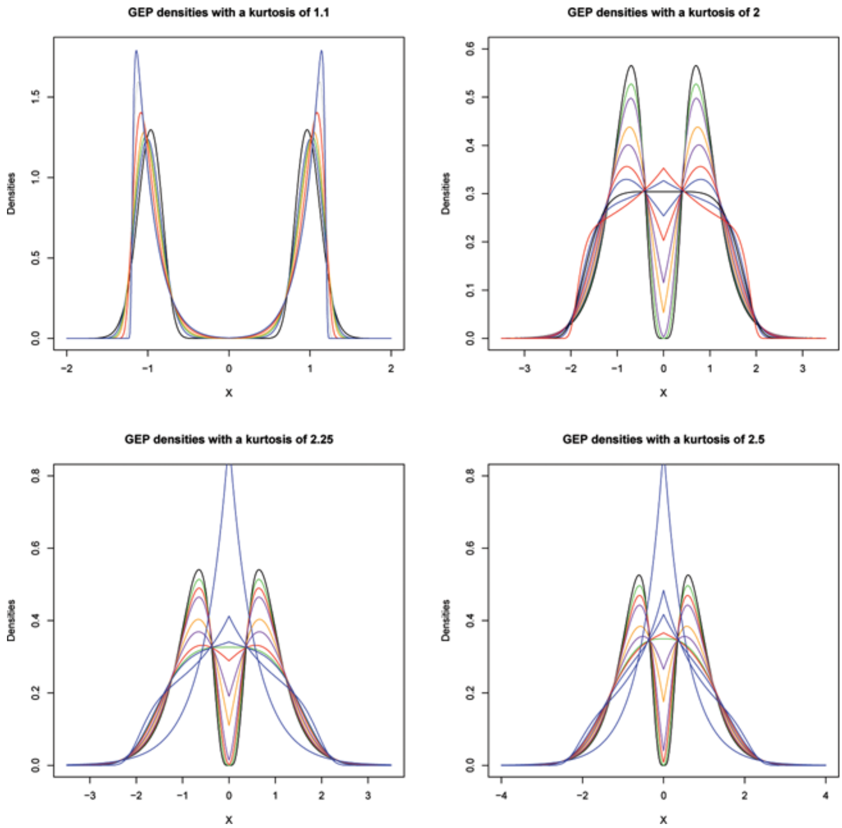

Figure 1. GEP densities with different tails behavior. (color figure available online.)

of 1.1, 2, 2.25, 2.5, 3, 4, 6, 10, we considered several GEP with different parameters $\theta_1, \theta_2, \theta_3$. For each kurtosis, we plotted these GEP densities in Figs. 1 and 2. It is interesting to see that for the same kurtosis, the tail behavior can vary significantly. The kurtosis have been chosen to cover the densities from very short-tailed to very heavy-tailed, and the parameters have been chosen to cover all the possible shapes for each given kurtosis. We think that this set of alternatives is very comprehensive and will give us the best indication of the performance of the tests to detect non-normality for symmetric alternatives.

We also studied 28 symmetric alternatives considered in Romão et al. (2010) and usually in the empirical power studies of tests for normality. They are listed in Tables 5 and 6. We find the Beta, Location Contaminated Normal (*LoConN*), Tukey, Johnson SB (*JSB*), Johnson SU (*JSU*), Student, logistic, and Laplace, with their kurtosis β_2 ranging from 1.5 to infinity. This set of alternatives is relatively limited when covering the full spectrum of tail behaviors, but it is still interesting to observe the performance of our test against the usual chosen alternative in empirical studies not using GEP densities. Note also that we reparametrized the classical Johnson SU distribution $JSU(\gamma, \delta, \xi, \lambda)$ to obtain our $JSU(v, \tau, \mu, \sigma)$ distribution by letting $v = -\gamma$, $\tau = 1/\delta$, $\mu = \xi - \lambda\sqrt{\exp(\delta^{-2})}\sinh(\gamma/\delta)$ and $\sigma = \lambda/c$. Doing so, the mean and standard deviation of our *JSU* distribution is respectively μ and σ; see Johnson (1949) for further details.

Table 3
Power against GEP distribution for 5% level tests, $M = 100,000$, $n = 20$, and $n = 50$

β_2	Alternative	$AD*$	Z_C	K^2	JB	T_w	$T_{K,5}$	W	CS	$BCMR$	β_3^2	R_{sJ}	S	R_n
1.10	GEP(1,−40.5,0)	100	100	89.7	4.2	96.0	84.8	100	100	100	90.0	54.4	99.8	94.4
1.10	GEP(4,−18.1,0)	100	100	89.9	4.1	96.1	84.8	100	100	100	91.3	54.2	99.9	93.3
1.10	GEP(6,−14.6,0)	100	100	89.8	4.2	95.9	84.6	100	100	100	92.0	54.9	100	92.8
1.10	GEP(8,−12.9,0)	100	100	89.9	4.1	95.9	84.8	100	100	100	92.5	55.3	100	92.5
1.10	GEP(15,−10.9,0)	100	100	90.1	4.1	95.6	84.5	100	100	100	93.8	56.8	100	92.2
1.10	GEP(30,−10,0)	100	100	90.3	4.0	95.5	84.4	100	100	100	94.5	57.6	100	92.1
1.10	GEP(60,−9.8,0)	100	100	90.5	4.0	95.5	84.3	100	100	100	94.8	57.7	100	92.1
2.00	GEP(0.5,−8.9,0)	63.3	39.3	24.6	1.7	64.1	36.7	46.5	49.7	43.5	23.2	38.2	27.5	53.2
2.00	GEP(0.8,−5.1,0)	52.9	32.8	20.8	1.5	58.6	32.7	38.4	41.5	35.5	21.3	36.2	24.8	45.1
2.00	GEP(1,−4.2,0)	44.8	28.1	18.0	1.3	53.3	29.6	32.4	35.5	29.5	20.0	34.4	23.3	38.7
2.00	GEP(1.5,−3.2,0)	29.6	20.5	13.2	1.0	40.6	23.8	22.6	25.1	20.0	17.8	29.5	20.6	26.1
2.00	GEP(2,−2.5,0)	22.2	16.5	10.9	0.8	33.0	20.5	17.5	19.6	15.2	16.6	26.1	19.3	20.4
2.00	GEP(3,−1.4,0)	15.3	12.9	8.5	0.6	24.1	16.6	12.9	14.7	11.0	15.4	20.8	18.2	15.3
2.00	GEP(4,−0.7,0)	12.2	11.2	7.0	0.5	19.7	14.6	10.8	12.4	9.1	14.4	17.7	17.6	13.1
2.00	GEP(6,0,0)	9.7	9.8	5.9	0.4	15.8	12.3	9.1	10.5	7.5	13.7	14.9	17.5	11.3
2.00	GEP(8,0.3,0)	8.8	9.5	5.6	0.4	14.3	11.7	8.6	9.9	7.0	13.7	13.5	17.9	10.8
2.00	GEP(15,0.7,0)	7.6	9.0	5.0	0.3	12.1	10.4	7.8	9.1	6.3	13.6	11.7	18.7	10.1
2.00	GEP(0,865,873)	27.4	25.1	28.5	30.5	27.4	28.2	26.1	25.1	27.8	30.0	29.2	28.2	26.0
2.25	GEP(0.5,−7.2,0)	45.5	26.6	16.0	2.4	50.4	25.8	31.8	34.4	29.4	15.7	31.3	18.5	39.9
2.25	GEP(0.7,−4.7,0)	38.0	22.7	13.5	2.1	45.1	22.7	26.6	29.0	24.5	14.4	28.9	17.0	33.1
2.25	GEP(0.8,−3.8,0)	31.8	19.5	11.9	1.9	40.4	20.3	22.5	24.7	20.4	13.4	27.2	15.8	28.0
2.25	GEP(1,−3.3,0)	26.0	16.5	10.1	1.8	35.0	17.9	18.6	20.7	16.8	12.4	25.0	14.7	22.8
2.25	GEP(1.5,−2.4,0)	15.1	10.8	6.6	1.4	23.0	12.7	11.5	12.9	10.2	10.0	18.8	12.0	13.5
2.25	GEP(2,−1.7,0)	10.7	8.2	5.1	1.1	16.9	10.1	8.5	9.6	7.4	8.8	14.8	10.8	10.1
2.25	GEP(3,−0.5,0)	7.2	6.1	3.6	1.0	11.2	7.5	6.1	6.9	5.2	7.3	10.6	9.2	7.4
2.25	GEP(3.7,0,0)	6.1	5.3	3.1	0.8	9.4	6.5	5.2	5.9	4.4	6.7	9.2	8.8	6.7
2.25	GEP(4,0.2,0)	5.6	5.2	2.9	0.8	8.7	6.1	5.0	5.7	4.2	6.5	8.5	8.7	6.6
2.25	GEP(8,1.2,0)	4.4	4.0	2.1	0.6	6.2	4.6	3.9	4.4	3.2	5.7	6.3	8.3	6.1
2.25	GEP(0,785,782)	27.5	25.1	28.6	30.6	27.5	28.5	26.1	25.0	28.0	30.4	29.3	28.3	26.2
2.50	GEP(0.5,−6.1,0)	33.7	19.8	11.8	3.2	38.7	18.9	23.4	25.4	21.7	11.7	25.1	13.9	29.5
2.50	GEP(0.7,−3.9,0)	26.1	16.0	9.8	3.2	32.5	15.9	18.4	20.1	16.9	10.2	22.5	12.0	23.1
2.50	GEP(0.8,−3.2,0)	20.7	13.4	8.3	2.8	27.3	13.6	15.0	16.5	13.7	9.2	20.0	10.9	18.3
2.50	GEP(1,−2.7,0)	16.0	11.0	6.9	2.7	22.2	11.6	12.0	13.2	10.9	8.3	17.3	9.7	13.9
2.50	GEP(1.5,−1.8,0)	8.8	7.0	4.5	2.2	12.6	7.6	7.2	8.0	6.4	6.2	11.3	7.6	7.7
2.50	GEP(2,−1,0)	6.3	5.4	3.4	1.9	8.9	5.6	5.4	6.0	4.9	5.2	8.4	6.5	5.9
2.50	GEP(2.8,0,0)	4.6	4.1	2.6	1.6	6.3	4.3	4.1	4.5	3.7	4.4	6.4	5.7	5.0
2.50	GEP(3,0.2,0)	4.4	3.7	2.4	1.6	5.8	4.0	3.8	4.1	3.3	4.2	5.8	5.5	4.8
2.50	GEP(4,0.9,0)	4.0	3.3	2.1	1.5	4.9	3.4	3.4	3.7	3.1	4.0	4.9	5.4	4.9
2.50	GEP(8,1.9,0)	3.8	2.7	1.6	1.2	3.9	2.5	2.9	3.0	2.7	3.6	3.9	5.1	5.4
2.50	GEP(0,730,739)	27.3	24.9	28.3	30.3	27.4	28.1	25.9	25.0	27.7	30.0	29.1	28.1	26.0
3.00	GEP(0.5,−4.6,0)	20.1	13.9	9.3	5.8	23.2	12.5	15.2	16.4	14.5	7.9	17.0	8.6	18.8
3.00	GEP(0.6,−3.6,0)	17.1	12.3	8.4	5.6	20.1	11.2	13.3	14.3	12.7	7.2	15.6	7.9	16.1
3.00	GEP(0.7,−2.9,0)	14.3	11.1	7.8	5.7	17.0	10.1	11.6	12.5	11.2	6.7	13.8	7.1	13.4
3.00	GEP(0.8,−2.5,0)	11.7	9.7	7.2	5.6	13.8	8.9	9.8	10.5	9.5	6.2	11.8	6.5	10.8
3.00	GEP(1,−1.9,0)	8.3	7.8	6.4	5.5	9.5	7.3	7.6	8.1	7.4	5.6	8.9	5.7	7.3
3.00	GEP(1.5,−1,0)	5.4	5.7	5.4	5.2	5.7	5.6	5.4	5.6	5.4	5.0	5.8	4.9	5.2
3.00	GEP(2,0,0)	5.1	5.2	5.1	5.1	5.1	5.0	5.0	5.0	5.0	5.2	5.2	5.1	5.2
3.00	GEP(3,1.4,0)	5.4	4.6	4.6	4.9	5.1	4.5	4.7	4.6	4.8	5.6	5.2	5.5	5.8
3.00	GEP(0,665,661)	27.5	25.1	28.7	30.7	27.4	28.2	26.1	25.1	28.0	30.0	29.1	28.2	26.0
4.00	GEP(0.4,−4.5,0)	14.7	14.0	12.2	11.4	13.5	12.4	13.9	14.4	14.1	8.4	12.1	7.2	15.3
4.00	GEP(0.5,−3.2,0)	13.1	13.1	12.0	11.6	11.7	11.8	12.8	13.1	13.1	8.4	11.0	7.0	13.4
4.00	GEP(0.6,−2.3,0)	11.6	12.6	12.1	12.0	10.1	11.6	12.0	12.2	12.4	8.6	10.1	7.0	12.0

(*continued*)

Table 3
Continued

β_2	Alternative	$AD*$	Z_C	K^2	JB	T_w	$T_{K,5}$	W	CS	$BCMR$	β_3^2	R_{sJ}	S	R_n
4.00	GEP(0.7,−1.9,0)	10.1	11.7	12.0	12.3	8.6	11.2	11.1	11.1	11.5	8.8	8.9	6.9	10.2
4.00	GEP(0.8,−1.5,0)	9.4	11.4	12.2	12.7	8.2	11.2	10.5	10.4	11.1	9.4	8.6	7.5	9.4
4.00	GEP(1,−1.1,0)	9.3	11.1	12.5	13.3	8.6	11.5	10.4	10.2	11.0	10.5	8.9	8.5	9.3
4.00	GEP(1.4,0,0)	11.2	11.5	13.5	14.5	10.8	12.7	11.3	10.9	12.1	12.9	11.1	11.2	11.0
4.00	GEP(2,1.5,0)	12.9	11.9	14.0	15.2	13.0	13.4	12.1	11.6	13.1	15.0	13.5	13.5	12.7
4.00	GEP(3,3.1,0)	15.5	12.8	14.9	16.3	15.5	14.6	13.6	12.9	14.8	17.5	16.2	16.3	15.2
4.00	GEP(0,580,584)	27.5	25.2	28.6	30.6	27.6	28.4	26.1	25.2	28.0	30.3	29.3	28.3	26.2
6.00	GEP(0.4,−2.9,0)	17.0	19.4	21.0	21.9	13.9	19.6	18.5	18.3	19.5	16.5	14.7	13.0	18.0
6.00	GEP(0.5,−1.9,0)	17.6	19.9	21.9	23.1	14.9	20.5	19.1	18.7	20.2	18.2	15.9	14.7	17.9
6.00	GEP(0.6,−1.3,0)	18.6	20.7	23.4	24.8	17.1	22.1	20.0	19.5	21.4	20.7	17.8	17.3	18.5
6.00	GEP(0.8,−0.8,0)	22.2	22.7	25.9	27.5	21.8	25.1	22.6	21.9	24.2	25.1	23.0	22.4	21.4
6.00	GEP(1,0,0)	27.3	24.9	28.5	30.4	27.3	28.1	25.8	24.9	27.6	30.0	29.1	28.1	25.9
6.00	GEP(2,3.9,0)	31.5	27.4	31.0	33.2	31.4	31.1	29.2	28.0	31.1	33.7	33.6	32.3	29.5
6.00	GEP(3,6.2,0)	33.5	28.9	32.4	34.8	33.4	32.8	30.9	29.6	33.0	35.8	35.8	34.2	31.2
6.00	GEP(0,140,129.1)	28.1	25.6	29.1	31.1	27.9	28.9	26.6	25.6	28.4	30.8	29.7	28.7	26.5
10.0	GEP(0.4,−1.7,0)	31.4	32.0	35.6	37.5	29.6	34.9	32.0	31.2	33.9	33.8	31.2	30.2	30.2
10.0	GEP(0.5,−1,0)	36.3	34.7	38.4	40.6	35.1	38.1	35.7	34.6	37.6	38.6	37.2	35.8	34.0
10.0	GEP(0.7,0,0)	49.1	41.7	44.7	47.4	47.8	45.9	45.1	43.7	47.4	49.5	51.9	48.6	45.5
10.0	GEP(1.4,5.7,0)	42.8	38.0	41.5	43.8	42.0	42.1	40.3	39.1	42.5	44.4	45.1	42.8	39.9
10.0	GEP(0,6,12.2)	38.8	34.7	38.3	40.6	38.5	38.7	36.5	35.3	38.6	40.9	41.2	39.3	36.5
10.0	GEP(0,17,−12.6)	39.2	35.2	38.9	41.1	38.8	39.2	37.1	35.9	39.2	41.3	41.5	39.6	36.8
Mean	$n = 20$	28.1	24.8	22.5	11.9	29.8	24.5	25.6	26.1	25.6	23.9	23.5	24.9	26.3
Rank		2	8	12	13	1	9	5.5	4	5.5	10	11	7	3
Dev. to the Best – Mean		3.8	7.0	9.4	20.0	2.1	7.4	6.2	5.8	6.3	8.0	8.4	7.0	5.6
Rank		2	7.5	12	13	1	9	5	4	6	10	11	7.5	3
Dev. to the Best – Max		11.1	25.8	39.5	96.0	8.2	27.4	20.9	17.8	23.8	40.9	45.8	36.6	14.6
Rank		2	7	10	13	1	8	5	4	6	11	12	9	3
1.10	GEP(1,−40.5,0)	100	100	37.1	100	100	99.5	100	100	100	100	83.3	100	100
1.10	GEP(4,−18.1,0)	100	100	37.0	100	100	99.5	100	100	100	100	86.0	100	100
1.10	GEP(6,−14.6,0)	100	100	37.1	100	100	99.4	100	100	100	100	86.7	100	100
1.10	GEP(8,−12.9,0)	100	100	37.6	100	100	99.4	100	100	100	100	87.2	100	100
1.10	GEP(15,−10.9,0)	100	100	38.1	100	100	99.3	100	100	100	100	88.0	100	100
1.10	GEP(30,−10,0)	100	100	38.7	100	100	99.3	100	100	100	100	88.5	100	100
1.10	GEP(60,−9.8,0)	100	100	38.9	100	100	99.3	100	100	100	100	88.7	100	100
2.00	GEP(0.5,−8.9,0)	99.4	77.5	64.2	3.7	96.0	65.5	93.2	94.3	92.4	52.1	76.0	26.9	95.3
2.00	GEP(0.8,−5.1,0)	97.7	69.8	61.3	2.3	94.9	61.6	86.7	88.7	85.2	51.1	75.7	25.8	93.2
2.00	GEP(1,−4.2,0)	94.3	63.4	59.3	1.5	93.5	59.1	80.0	82.9	77.7	50.9	75.4	25.5	90.1
2.00	GEP(1.5,−3.2,0)	78.3	50.7	54.7	0.7	87.3	52.5	62.4	67.0	58.6	50.9	73.4	25.4	75.1
2.00	GEP(2,−2.5,0)	63.4	44.0	51.7	0.4	79.1	47.8	51.5	56.9	47.0	51.0	69.0	26.4	60.5
2.00	GEP(3,−1.4,0)	45.7	38.1	47.8	0.2	64.9	42.1	40.6	46.4	35.6	51.9	59.7	28.7	44.1
2.00	GEP(4,−0.7,0)	37.0	36.2	45.5	0.1	54.6	38.5	36.0	42.1	30.9	52.8	51.8	31.7	36.7
2.00	GEP(6,0,0)	29.2	36.5	43.6	0.1	43.9	35.0	33.2	39.6	27.7	54.5	42.8	37.4	30.4
2.00	GEP(8,0.3,0)	26.4	37.5	42.6	0.1	38.5	33.3	32.7	39.4	26.9	55.8	38.0	42.2	28.0
2.00	GEP(15,0.7,0)	23.4	41.3	41.2	0.0	32.4	30.9	34.0	41.0	27.5	58.1	32.7	51.0	25.4
2.00	GEP(0,865,873)	55.0	46.0	49.1	55.4	62.8	59.3	52.5	48.8	54.2	60.9	63.8	69.4	58.2
2.25	GEP(0.5,−7.2,0)	94.9	55.3	44.1	2.6	89.0	45.0	77.0	79.4	75.3	33.9	67.3	15.6	90.8
2.25	GEP(0.7,−4.7,0)	89.4	47.5	40.0	2.1	86.2	40.0	67.4	70.7	65.1	31.7	65.3	14.3	86.9
2.25	GEP(0.8,−3.8,0)	81.6	41.2	37.1	1.6	82.9	36.6	58.6	62.4	55.7	30.5	64.0	13.6	81.2
2.25	GEP(1,−3.3,0)	71.0	35.0	34.1	1.4	78.1	33.0	49.0	53.2	45.9	29.3	61.6	12.8	72.0
2.25	GEP(1.5,−2.4,0)	42.0	23.1	26.6	0.7	58.8	24.1	29.3	33.7	26.0	26.6	50.4	11.8	40.9
2.25	GEP(2,−1.7,0)	27.9	17.4	22.1	0.4	44.5	19.0	20.7	24.5	17.7	24.3	40.6	11.3	26.1
2.25	GEP(3,−0.5,0)	16.0	13.3	17.3	0.2	28.0	13.9	13.8	17.2	11.2	22.3	27.2	11.4	15.5

(*continued*)

Table 3
Continued

β_2	Alternative	$AD*$	Z_C	K^2	JB	T_w	$T_{K,5}$	W	CS	$BCMR$	β_3^2	R_{sJ}	S	R_n
2.25	GEP(3.7,0,0)	13.1	12.4	15.9	0.1	22.6	12.2	12.3	15.5	9.7	21.5	22.5	11.7	13.2
2.25	GEP(4,0.2,0)	11.8	12.0	15.3	0.1	20.6	11.6	11.5	14.7	9.1	21.2	20.5	12.0	12.1
2.25	GEP(8,1.2,0)	7.6	11.3	12.0	0.0	11.3	8.3	9.4	12.4	7.0	20.5	11.9	15.1	9.9
2.25	GEP(0,785,782)	54.8	46.2	49.5	55.6	63.2	59.6	52.5	48.9	54.4	61.3	64.1	69.8	58.5
2.50	GEP(0.5,−6.1,0)	83.7	38.4	29.6	3.7	77.2	29.8	58.0	61.1	56.1	21.8	56.5	9.2	84.2
2.50	GEP(0.7,−3.9,0)	70.8	30.7	25.5	3.1	70.7	25.0	45.9	49.2	43.6	19.6	52.5	8.3	75.3
2.50	GEP(0.8,−3.2,0)	56.8	24.6	21.8	2.7	63.1	20.8	35.8	39.1	33.5	17.8	48.4	7.5	62.7
2.50	GEP(1,−2.7,0)	42.9	19.5	18.6	2.3	53.4	17.2	27.1	30.2	24.8	16.3	42.8	6.9	46.5
2.50	GEP(1.5,−1.8,0)	18.8	10.8	11.7	1.5	29.6	10.1	13.2	15.5	11.5	12.4	26.7	5.4	17.8
2.50	GEP(2,−1,0)	11.0	7.6	8.7	1.0	18.7	6.9	8.6	10.5	7.1	10.6	17.9	5.1	9.8
2.50	GEP(2.8,0,0)	6.8	5.5	6.2	0.6	10.8	4.5	5.9	7.4	4.7	8.6	10.9	4.8	6.1
2.50	GEP(3,0.2,0)	6.1	5.1	5.6	0.6	9.6	4.1	5.3	6.7	4.2	8.4	9.7	4.8	5.8
2.50	GEP(4,0.9,0)	4.8	4.2	4.4	0.4	6.5	3.1	4.2	5.5	3.3	7.3	6.8	5.0	5.4
2.50	GEP(8,1.9,0)	4.3	3.6	2.7	0.2	3.7	1.7	3.8	4.7	2.8	5.8	3.8	6.1	6.9
2.50	GEP(0,730,739)	54.9	45.7	48.8	55.0	62.8	59.2	52.4	48.7	54.1	61.0	63.8	69.4	58.4
3.00	GEP(0.5,−4.6,0)	53.2	23.0	17.3	8.0	48.6	16.9	33.0	34.9	31.9	11.0	35.2	4.1	67.1
3.00	GEP(0.6,−3.6,0)	44.3	20.3	15.7	7.9	42.7	15.2	27.9	29.7	26.9	10.1	31.7	3.7	58.9
3.00	GEP(0.7,−2.9,0)	34.4	17.0	13.7	7.5	35.9	13.2	22.3	23.9	21.4	9.2	27.8	3.4	47.3
3.00	GEP(0.8,−2.5,0)	25.1	14.4	12.2	7.3	28.3	11.7	17.6	18.9	17.0	8.4	22.9	3.3	34.6
3.00	GEP(1,−1.9,0)	13.2	10.3	9.3	6.6	16.2	8.7	11.2	12.0	10.8	6.8	14.5	2.9	16.5
3.00	GEP(1.5,−1,0)	6.0	6.6	6.3	5.6	6.5	6.1	6.3	6.6	6.1	5.3	6.5	3.6	6.4
3.00	GEP(2,0,0)	5.0	5.0	5.0	4.8	5.0	4.9	5.1	5.1	5.0	5.2	5.1	5.1	5.1
3.00	GEP(3,1.4,0)	5.8	3.7	3.5	3.8	5.7	3.9	4.6	4.3	4.4	5.0	5.7	8.2	6.5
3.00	GEP(0,665,661)	54.7	46.0	49.0	55.2	62.8	59.3	52.6	48.9	54.4	60.9	63.7	69.2	58.4
4.00	GEP(0.4,−4.5,0)	30.1	23.1	19.9	19.4	20.9	20.5	25.5	25.3	26.1	13.0	17.5	5.4	51.0
4.00	GEP(0.5,−3.2,0)	24.1	21.4	19.3	19.6	16.7	20.1	22.8	22.3	23.4	13.3	14.8	5.8	42.6
4.00	GEP(0.6,−2.3,0)	18.9	20.4	19.4	20.4	13.0	20.5	20.8	20.1	21.5	14.1	12.5	6.9	33.8
4.00	GEP(0.7,−1.9,0)	15.1	19.3	19.4	21.0	10.9	20.8	18.8	17.9	19.7	15.1	11.1	8.9	25.5
4.00	GEP(0.8,−1.5,0)	13.3	18.7	19.6	21.7	10.7	21.5	17.8	16.7	18.7	16.4	11.0	11.5	20.8
4.00	GEP(1,−1.1,0)	12.9	17.6	19.3	21.9	12.9	22.0	16.9	15.5	17.8	18.4	13.2	16.3	18.3
4.00	GEP(1.4,0,0)	17.4	17.4	19.9	23.3	21.6	24.5	18.5	16.5	19.6	23.6	21.6	27.6	21.5
4.00	GEP(2,1.5,0)	22.8	17.9	20.5	24.4	28.8	26.4	21.2	18.6	22.3	27.6	29.0	36.2	26.2
4.00	GEP(3,3.1,0)	30.0	18.2	20.8	25.4	37.3	28.9	24.6	21.3	25.8	32.6	37.8	45.6	32.5
4.00	GEP(0,580,584)	54.8	45.8	48.9	55.3	62.8	59.3	52.6	48.8	54.2	60.8	63.8	69.3	58.2
6.00	GEP(0.4,−2.9,0)	30.3	35.4	36.9	40.6	22.8	40.9	35.7	33.7	37.3	35.1	24.2	25.4	45.9
6.00	GEP(0.5,−1.9,0)	31.6	37.0	39.3	43.5	28.0	44.2	37.4	35.1	39.1	39.6	28.9	32.5	43.8
6.00	GEP(0.6,−1.3,0)	34.1	38.3	41.4	46.1	35.4	47.6	39.2	36.5	41.0	44.4	36.0	41.3	42.9
6.00	GEP(0.8,−0.8,0)	42.9	41.2	44.9	50.4	49.5	53.0	44.7	41.4	46.4	52.4	49.8	56.2	48.1
6.00	GEP(1,0,0)	54.5	45.6	48.7	54.8	62.8	59.0	52.2	48.4	54.0	60.8	63.8	69.4	57.9
6.00	GEP(2,3.9,0)	63.3	50.6	53.1	59.9	70.9	65.0	59.4	55.4	61.0	67.5	72.2	76.9	65.8
6.00	GEP(3,6.2,0)	67.0	53.1	55.3	62.5	74.3	67.8	62.6	58.5	64.2	70.7	75.6	79.7	69.0
6.00	GEP(0,140,129.1)	55.6	46.6	49.7	55.9	63.6	60.0	53.3	49.6	55.0	61.7	64.6	70.2	59.0
10.0	GEP(0.4,−1.7,0)	59.9	58.7	61.6	66.8	62.6	69.3	62.1	59.1	63.7	68.3	63.1	68.0	65.8
10.0	GEP(0.5,−1,0)	68.6	63.3	65.9	71.4	73.7	74.6	68.5	65.4	70.0	75.0	74.4	78.6	71.7
10.0	GEP(0.7,0,0)	86.0	73.8	74.0	79.7	89.3	83.9	82.0	79.3	83.1	86.1	90.8	92.1	86.3
10.0	GEP(1.4,5.7,0)	77.9	68.2	69.7	75.4	83.1	79.2	75.3	72.2	76.6	80.8	84.3	87.0	79.4
10.0	GEP(0,6,12.2)	73.3	63.6	65.5	71.3	79.3	75.3	70.6	67.4	72.1	76.9	80.3	83.7	75.2
10.0	GEP(0,17,−12.6)	73.7	64.3	66.1	71.9	79.7	75.9	71.2	68.0	72.6	77.4	80.8	84.0	75.9
Mean	$n = 50$	48.2	39.1	33.6	28.9	52.1	41.2	43.8	44.3	43.1	41.9	47.1	37.0	51.2
Rank		3	10	12	13	1	9	6	5	7	8	4	11	2
Dev. to the Best – Mean		9.7	18.8	24.3	29.0	5.8	16.7	14.1	13.6	14.8	16.1	10.8	20.9	6.7
Rank		3	10	12	13	1	9	6	5	7	8	4	11	2
Dev. to the Best – Max		34.7	45.8	63.0	95.7	30.1	54.4	34.1	32.2	35.2	62.4	33.5	79.3	32.7
Rank		6	8	11	13	1	9	5	2	7	10	4	12	3

Table 4
Power against GEP distribution for 5% level tests, $M = 100,000$, $n = 100$, and $n = 200$

β_2	Alternative	$AD*$	Z_C	K^2	JB	T_w	$T_{K,5}$	W	CS	$BCMR$	β_3^2	R_{sJ}	S	R_n
1.10	GEP(1,−40.5,0)	100	100	0.8	100	100	100	100	100	100	100	97.4	100	100
1.10	GEP(4,−18.1,0)	100	100	0.9	100	100	100	100	100	100	100	97.8	100	100
1.10	GEP(6,−14.6,0)	100	100	0.8	100	100	100	100	100	100	100	98.1	100	100
1.10	GEP(8,−12.9,0)	100	100	0.9	100	100	100	100	100	100	100	98.2	100	100
1.10	GEP(15,−10.9,0)	100	100	0.9	100	100	100	100	100	100	100	98.4	100	100
1.10	GEP(30,−10,0)	100	100	0.9	100	100	100	100	100	100	100	98.6	100	100
1.10	GEP(60,−9.8,0)	100	100	0.9	100	100	100	100	100	100	100	98.7	100	100
2.00	GEP(0.5,−8.9,0)	100	97.8	85.0	52.9	99.9	90.7	100	100	100	76.9	95.5	15.1	99.9
2.00	GEP(0.8,−5.1,0)	100	95.2	85.4	48.8	99.9	89.5	99.8	99.8	99.8	78.0	95.8	15.0	99.8
2.00	GEP(1,−4.2,0)	100	92.2	86.0	45.2	99.8	88.8	99.3	99.4	99.2	79.5	96.1	15.3	99.7
2.00	GEP(1.5,−3.2,0)	99.0	84.1	86.7	38.4	99.5	86.7	94.6	95.8	93.8	82.8	96.3	17.0	98.4
2.00	GEP(2,−2.5,0)	95.2	79.0	87.4	33.8	98.6	84.9	88.3	91.0	86.6	85.3	95.6	19.5	94.2
2.00	GEP(3,−1.4,0)	84.2	75.8	88.5	28.8	94.5	82.8	79.9	84.6	76.7	88.9	92.2	25.2	82.7
2.00	GEP(4,−0.7,0)	75.1	76.5	88.6	26.0	88.3	80.9	76.3	82.1	72.4	90.6	86.8	32.0	74.2
2.00	GEP(6,0,0)	65.4	80.3	88.8	23.0	78.3	78.6	75.6	82.1	70.7	92.8	77.7	44.6	65.3
2.00	GEP(8,0.3,0)	62.0	84.2	88.9	22.2	72.3	77.4	77.6	84.2	72.4	94.2	72.2	55.5	62.2
2.00	GEP(15,0.7,0)	58.5	90.2	88.6	20.7	63.6	75.3	82.3	88.5	76.8	95.7	64.1	74.3	58.4
2.00	GEP(0,865,873)	82.7	69.6	72.5	80.2	90.4	84.8	80.0	76.2	81.0	86.2	91.0	94.1	85.7
2.25	GEP(0.5,−7.2,0)	100	84.8	64.4	25.6	99.3	71.2	99.0	99.1	99.0	52.8	91.7	6.2	99.6
2.25	GEP(0.7,−4.7,0)	99.9	77.9	62.3	21.6	98.9	67.5	97.0	97.3	96.9	51.9	91.3	5.9	99.3
2.25	GEP(0.8,−3.8,0)	99.6	70.3	60.0	18.1	98.6	63.6	93.1	94.0	92.7	51.0	91.0	5.4	98.9
2.25	GEP(1,−3.3,0)	98.0	62.6	58.5	15.3	97.7	60.3	86.1	88.0	85.2	51.0	90.3	5.4	97.6
2.25	GEP(1.5,−2.4,0)	79.1	44.2	53.0	8.7	90.2	49.8	59.9	64.7	57.4	50.4	84.6	5.4	79.9
2.25	GEP(2,−1.7,0)	58.4	36.1	49.6	5.7	78.3	43.4	44.7	50.8	41.4	50.1	74.7	5.6	57.0
2.25	GEP(3,−0.5,0)	35.7	31.2	45.9	3.5	56.1	35.7	32.5	39.2	28.7	50.6	55.2	6.9	34.1
2.25	GEP(3.7,0,0)	28.5	30.3	43.9	2.8	45.5	32.4	29.1	36.1	25.1	50.7	45.3	7.6	28.1
2.25	GEP(4,0.2,0)	26.0	30.5	43.4	2.5	41.4	31.2	28.3	35.4	24.2	51.0	41.5	8.2	26.0
2.25	GEP(8,1.2,0)	16.8	37.3	40.2	1.4	22.0	25.1	28.7	37.4	23.3	54.6	22.7	15.3	21.1
2.25	GEP(0,785,782)	82.7	69.6	72.5	80.1	90.2	84.7	79.8	76.0	80.9	86.1	90.8	94.0	85.6
2.50	GEP(0.5,−6.1,0)	99.8	64.3	44.1	13.5	96.1	49.2	92.9	93.3	92.8	32.9	84.6	2.7	98.8
2.50	GEP(0.7,−3.9,0)	98.3	52.2	39.4	9.9	94.0	42.2	82.7	84.0	82.2	30.3	82.2	2.2	97.7
2.50	GEP(0.8,−3.2,0)	92.7	41.8	35.4	7.6	90.6	36.4	69.3	71.7	68.3	28.4	79.4	2.1	95.0
2.50	GEP(1,−2.7,0)	80.5	32.7	31.6	5.6	84.2	30.8	53.8	57.2	52.3	26.8	74.5	1.8	86.5
2.50	GEP(1.5,−1.8,0)	39.1	17.6	23.2	2.5	56.8	19.0	24.8	28.7	22.8	22.9	52.9	1.5	40.4
2.50	GEP(2,−1,0)	21.2	12.5	18.2	1.3	36.5	13.3	15.0	18.6	13.1	20.2	35.5	1.5	19.2
2.50	GEP(2.8,0,0)	11.2	9.4	14.1	0.7	19.6	9.0	9.7	12.7	8.0	17.9	19.8	1.6	9.8
2.50	GEP(3,0.2,0)	9.9	9.1	13.5	0.6	17.1	8.2	8.9	12.1	7.3	17.6	17.3	1.7	8.9
2.50	GEP(4,0.9,0)	6.9	8.5	11.2	0.3	9.9	6.2	7.5	10.6	5.9	16.3	10.2	2.1	7.9
2.50	GEP(8,1.9,0)	6.2	10.1	8.0	0.1	4.0	3.5	8.0	11.5	5.7	15.1	4.3	4.1	12.0
2.50	GEP(0,730,739)	82.7	69.4	72.3	80.0	90.3	84.7	79.8	75.9	80.8	86.2	90.9	94.0	85.5
3.00	GEP(0.5,−4.6,0)	91.4	35.7	23.0	12.1	74.7	24.6	63.4	64.0	63.6	13.5	59.6	0.7	95.4
3.00	GEP(0.6,−3.6,0)	83.6	30.1	20.5	11.3	68.6	21.4	53.3	54.0	53.4	12.2	55.1	0.6	92.7
3.00	GEP(0.7,−2.9,0)	69.9	24.8	17.9	10.4	59.7	18.2	41.6	42.5	41.6	10.7	48.5	0.6	86.5
3.00	GEP(0.8,−2.5,0)	52.2	20.1	15.7	9.7	48.9	15.6	30.7	31.7	30.7	9.9	41.0	0.6	73.6
3.00	GEP(1,−1.9,0)	23.8	13.3	11.5	8.3	27.4	11.1	16.4	17.1	16.4	7.7	24.8	0.7	37.0
3.00	GEP(1.5,−1,0)	6.9	7.2	7.2	6.3	7.9	6.8	7.0	7.2	6.9	5.8	8.0	2.4	8.2
3.00	GEP(2,0,0)	5.1	4.9	5.0	5.1	5.0	5.0	4.9	5.0	4.9	5.0	5.2	5.1	5.0
3.00	GEP(3,1.4,0)	6.8	3.0	3.1	3.5	7.0	3.4	4.5	4.3	4.2	4.5	7.2	11.7	8.0
3.00	GEP(0,665,661)	82.8	69.5	72.4	80.2	90.3	84.7	79.8	76.0	80.9	86.1	90.9	94.1	85.6
4.00	GEP(0.4,−4.5,0)	60.3	33.4	27.6	29.3	28.7	30.5	43.1	41.1	44.5	19.4	23.4	3.9	87.4
4.00	GEP(0.5,−3.2,0)	47.0	32.0	27.8	30.3	21.6	30.9	37.7	35.6	39.2	20.9	18.8	5.1	80.7

(*continued*)

Table 4
Continued

β_2	Alternative	$AD*$	Z_C	K^2	JB	T_w	$T_{K,5}$	W	CS	$BCMR$	β_3^2	R_{sJ}	S	R_n
4.00	GEP(0.6,−2.3,0)	33.5	30.2	28.1	31.4	15.7	31.8	33.0	30.6	34.5	22.5	14.8	6.9	68.6
4.00	GEP(0.7,−1.9,0)	23.2	28.3	28.2	32.2	12.1	32.4	28.7	26.2	30.3	24.3	12.7	10.6	50.5
4.00	GEP(0.8,−1.5,0)	18.4	26.9	28.4	33.1	13.1	33.4	26.6	23.8	28.1	26.7	13.9	16.2	37.0
4.00	GEP(1,−1.1,0)	18.1	25.9	28.9	34.3	19.4	35.1	25.8	22.6	27.3	30.4	20.0	26.4	30.1
4.00	GEP(1.4,0,0)	28.3	25.3	29.6	36.3	38.4	39.1	29.4	25.5	30.8	38.6	38.9	49.4	34.8
4.00	GEP(2,1.5,0)	39.5	25.6	30.0	38.0	52.5	42.7	34.7	30.0	35.9	44.6	53.1	63.9	44.4
4.00	GEP(3,3.1,0)	54.1	27.1	30.5	39.9	66.7	47.7	43.0	37.6	44.0	51.9	67.8	76.7	57.2
4.00	GEP(0,580,584)	83.0	69.8	72.6	80.2	90.4	84.9	80.1	76.3	81.1	86.4	91.0	94.0	85.8
6.00	GEP(0.4,−2.9,0)	50.5	54.7	56.7	62.8	33.5	64.1	57.5	53.6	59.5	59.0	36.4	39.9	78.0
6.00	GEP(0.5,−1.9,0)	51.3	56.5	59.6	65.9	43.5	67.9	59.3	55.2	61.1	64.6	45.6	51.6	72.9
6.00	GEP(0.6,−1.3,0)	55.5	58.5	62.6	69.7	57.8	72.0	62.3	57.9	64.0	70.2	58.8	66.2	69.1
6.00	GEP(0.8,−0.8,0)	68.6	63.0	67.1	74.6	78.1	78.4	70.1	65.6	71.4	78.9	78.6	84.6	74.7
6.00	GEP(1,0,0)	82.6	69.1	72.2	80.0	90.1	84.5	79.7	75.8	80.7	85.9	90.7	93.9	85.5
6.00	GEP(2,3.9,0)	90.2	75.9	77.4	84.8	95.1	89.4	86.6	83.5	87.3	91.0	95.5	97.2	91.6
6.00	GEP(3,6.2,0)	92.4	79.3	80.3	87.3	96.3	91.7	89.4	86.8	90.0	92.9	96.7	98.0	93.4
6.00	GEP(0,140,129.1)	83.8	70.9	73.6	81.1	91.1	85.5	81.1	77.3	82.1	86.9	91.6	94.5	86.5
10.0	GEP(0.4,−1.7,0)	85.7	82.9	85.2	89.7	87.9	91.7	87.3	84.8	88.1	91.6	88.4	91.6	90.8
10.0	GEP(0.5,−1,0)	91.9	87.0	88.4	92.5	95.1	94.5	91.8	89.8	92.3	94.9	95.3	97.0	93.8
10.0	GEP(0.7,0,0)	99.0	94.2	93.7	96.5	99.6	98.3	98.0	97.4	98.2	98.7	99.7	99.8	99.1
10.0	GEP(1.4,5.7,0)	96.8	90.9	91.4	94.9	98.6	96.7	95.7	94.4	96.0	97.2	98.7	99.3	97.4
10.0	GEP(0,6,12.2)	95.0	87.6	88.5	92.9	97.7	95.2	93.5	91.8	94.0	95.8	97.9	98.7	95.9
10.0	GEP(0,17,−12.6)	95.3	88.1	88.9	93.1	97.7	95.5	93.8	92.2	94.2	96.1	98.0	98.8	96.1
Mean	$n = 100$	66.6	56.4	47.3	42.8	68.7	58.8	63.2	63.5	62.7	58.8	66.4	42.0	71.3
Rank		3	10	11	12	2	8.5	6	5	7	8.5	4	13	1
Dev. to the Best – Mean		11.4	21.6	30.7	35.2	9.3	19.2	14.8	14.5	15.2	19.2	11.6	35.9	6.7
Rank		3	10	11	12	2	8.5	6	5	7	8.5	4	13	1
Dev. to the Best – Max		37.8	62.6	99.2	88.4	59.1	71.3	44.9	46.3	44.9	81.9	64.0	97.1	37.3
Rank		2	7	13	11	6	9	3.5	5	3.5	10	8	12	1
1.10	GEP(1,−40.5,0)	100	100	0.0	100	100	100	100	100	100	100	99.9	100	100
1.10	GEP(4,−18.1,0)	100	100	0.0	100	100	100	100	100	100	100	99.9	100	100
1.10	GEP(6,−14.6,0)	100	100	0.0	100	100	100	100	100	100	100	100	100	100
1.10	GEP(8,−12.9,0)	100	100	0.0	100	100	100	100	100	100	100	100	100	100
1.10	GEP(15,−10.9,0)	100	100	0.0	100	100	100	100	100	100	100	100	100	100
1.10	GEP(30,−10,0)	100	100	0.0	100	100	100	100	100	100	100	100	100	100
1.10	GEP(60,−9.8,0)	100	100	0.0	100	100	100	100	100	100	100	100	100	100
2.00	GEP(0.5,−8.9,0)	100	100	95.5	89.0	100	99.5	100	100	100	93.9	99.9	3.8	100
2.00	GEP(0.8,−5.1,0)	100	100	96.5	89.9	100	99.4	100	100	100	95.3	99.9	4.0	100
2.00	GEP(1,−4.2,0)	100	99.9	97.3	90.8	100	99.4	100	100	100	96.5	99.9	4.4	100
2.00	GEP(1.5,−3.2,0)	100	99.4	98.6	92.6	100	99.5	100	100	100	98.4	100	5.9	100
2.00	GEP(2,−2.5,0)	100	98.7	99.3	93.7	100	99.5	99.8	99.9	99.8	99.2	100	7.9	100
2.00	GEP(3,−1.4,0)	99.5	98.5	99.7	95.2	99.9	99.5	99.2	99.5	99.1	99.8	99.9	14.1	99.5
2.00	GEP(4,−0.7,0)	98.4	98.9	99.8	95.7	99.6	99.6	99.0	99.4	98.7	99.9	99.6	22.4	98.4
2.00	GEP(6,0,0)	96.7	99.6	99.9	96.1	98.2	99.6	99.3	99.7	99.0	100	98.2	42.1	97.0
2.00	GEP(8,0.3,0)	95.8	99.9	99.9	96.2	96.6	99.6	99.6	99.8	99.3	100	96.5	59.9	96.5
2.00	GEP(15,0.7,0)	95.7	100	99.9	96.1	92.8	99.4	99.9	100	99.8	100	92.9	88.9	96.2
2.00	GEP(0,865,873)	98.4	92.5	93.6	96.6	99.6	98.3	97.6	96.5	97.7	98.6	99.6	99.8	98.8
2.25	GEP(0.5,−7.2,0)	100	99.6	80.8	63.2	100	93.5	100	100	100	73.7	99.5	0.8	100
2.25	GEP(0.7,−4.7,0)	100	98.5	80.2	60.7	100	91.9	100	100	100	74.0	99.5	0.7	100
2.25	GEP(0.8,−3.8,0)	100	96.4	80.1	58.3	100	90.6	100	100	100	74.6	99.5	0.7	100
2.25	GEP(1,−3.3,0)	100	92.6	80.3	56.6	100	89.0	99.8	99.8	99.8	76.0	99.6	0.7	100
2.25	GEP(1.5,−2.4,0)	98.9	77.8	81.0	51.2	99.7	84.4	93.4	94.3	92.9	79.8	99.1	0.8	99.1
2.25	GEP(2,−1.7,0)	91.8	69.7	81.6	47.5	98.2	80.7	82.3	85.4	80.7	82.1	97.4	1.0	91.8
2.25	GEP(3,−0.5,0)	70.7	66.5	82.4	43.1	87.8	76.2	69.3	74.9	66.0	85.4	87.3	1.6	69.8

(*continued*)

Table 4
Continued

β_2	Alternative	$AD*$	Z_C	K^2	JB	T_w	$T_{K,5}$	W	CS	$BCMR$	β_3^2	R_{sJ}	S	R_n
2.25	GEP(3.7,0,0)	61.0	68.5	82.9	41.4	78.9	74.7	66.6	73.4	62.7	87.3	78.8	2.1	61.5
2.25	GEP(4,0.2,0)	56.9	70.0	83.1	40.4	74.1	74.0	66.2	73.5	61.9	88.1	74.1	2.4	58.5
2.25	GEP(8,1.2,0)	43.1	86.7	83.4	36.4	43.3	70.0	76.2	83.9	70.5	92.8	44.0	9.7	53.4
2.25	GEP(0,785,782)	98.5	92.7	93.9	96.7	99.6	98.3	97.6	96.6	97.8	98.5	99.6	99.8	98.9
2.50	GEP(0.5,−6.1,0)	100	94.0	58.0	36.6	99.9	75.7	100	100	100	46.8	98.2	0.2	100
2.50	GEP(0.7,−3.9,0)	100	84.4	54.0	30.9	99.8	68.8	99.7	99.7	99.7	44.9	97.9	0.1	100
2.50	GEP(0.8,−3.2,0)	100	72.5	51.1	26.7	99.5	62.7	97.8	97.7	97.8	43.9	97.4	0.1	99.9
2.50	GEP(1,−2.7,0)	99.3	58.7	47.7	22.4	98.8	56.1	90.4	90.5	90.2	42.6	96.4	0.1	99.6
2.50	GEP(1.5,−1.8,0)	73.4	32.8	40.2	13.9	87.3	39.8	51.0	54.4	49.3	40.3	84.8	0.1	77.2
2.50	GEP(2,−1,0)	43.7	24.6	36.0	10.0	66.0	31.2	31.6	36.2	29.4	39.1	64.9	0.1	41.9
2.50	GEP(2.8,0,0)	22.2	21.4	32.0	6.7	38.3	23.9	21.3	26.4	18.8	38.4	38.4	0.1	20.1
2.50	GEP(3,0.2,0)	18.8	21.1	31.0	6.1	32.9	22.6	19.8	25.2	17.3	38.4	33.1	0.2	17.7
2.50	GEP(4,0.9,0)	12.1	22.9	27.9	4.3	16.9	18.1	18.1	24.1	15.0	38.0	17.2	0.3	15.4
2.50	GEP(8,1.9,0)	12.1	39.7	24.1	2.6	4.8	12.9	28.8	38.2	22.7	41.8	5.0	1.8	28.7
2.50	GEP(0,730,739)	98.4	92.6	93.7	96.7	99.6	98.3	97.6	96.5	97.7	98.6	99.6	99.8	98.8
3.00	GEP(0.5,−4.6,0)	100	59.9	26.9	18.4	93.9	36.6	96.6	95.9	96.8	15.2	85.3	0.0	99.9
3.00	GEP(0.6,−3.6,0)	99.8	50.0	24.2	16.6	91.0	31.7	90.9	89.7	91.3	14.0	82.0	0.0	99.8
3.00	GEP(0.7,−2.9,0)	98.0	39.9	21.3	15.0	85.4	26.7	78.8	76.9	79.3	12.5	76.4	0.0	99.4
3.00	GEP(0.8,−2.5,0)	89.5	30.7	18.6	13.3	76.1	22.0	60.8	58.9	61.4	11.1	68.1	0.0	97.8
3.00	GEP(1,−1.9,0)	47.9	18.5	13.6	10.7	47.4	14.7	28.6	27.6	29.1	8.8	43.6	0.1	72.9
3.00	GEP(1.5,−1,0)	8.6	8.3	7.6	7.0	10.3	7.4	8.4	8.1	8.4	6.0	10.2	1.4	11.6
3.00	GEP(2,0,0)	5.0	5.2	5.1	5.2	4.9	5.0	5.2	5.0	5.1	4.9	5.0	4.9	4.8
3.00	GEP(3,1.4,0)	9.0	3.0	2.5	3.0	9.9	2.9	5.2	5.0	4.7	4.0	10.1	16.6	11.5
3.00	GEP(0,665,661)	98.4	92.8	93.9	96.7	99.6	98.3	97.7	96.6	97.8	98.6	99.6	99.8	98.9
4.00	GEP(0.4,−4.5,0)	95.6	52.0	38.8	42.8	39.9	45.1	76.1	71.6	77.4	31.0	32.2	2.5	99.3
4.00	GEP(0.5,−3.2,0)	86.0	48.8	39.7	44.5	28.4	46.0	66.0	60.6	67.7	33.5	23.8	3.6	98.4
4.00	GEP(0.6,−2.3,0)	66.1	45.5	40.6	45.9	18.3	47.2	55.5	50.0	57.4	36.6	16.8	6.0	95.6
4.00	GEP(0.7,−1.9,0)	42.1	42.4	41.3	47.4	12.8	48.6	47.0	41.4	48.9	40.1	13.5	11.2	83.8
4.00	GEP(0.8,−1.5,0)	29.3	40.3	42.4	49.0	15.9	50.6	42.1	36.5	44.0	44.1	17.1	20.9	63.2
4.00	GEP(1,−1.1,0)	28.2	38.7	43.4	51.2	29.9	53.6	40.9	35.1	42.7	50.0	30.7	39.5	48.3
4.00	GEP(1.4,0,0)	48.5	39.2	46.1	55.6	64.0	60.5	49.2	42.5	50.5	61.2	64.7	74.2	56.3
4.00	GEP(2,1.5,0)	67.9	41.9	48.0	59.0	81.7	66.8	60.4	53.7	61.3	69.0	82.3	88.6	72.1
4.00	GEP(3,3.1,0)	84.6	47.0	50.3	63.2	92.7	74.5	73.6	67.9	74.0	77.2	93.3	96.2	86.4
4.00	GEP(0,580,584)	98.4	92.7	93.8	96.8	99.6	98.3	97.7	96.7	97.8	98.6	99.6	99.8	98.9
6.00	GEP(0.4,−2.9,0)	80.7	78.5	79.8	84.9	48.8	86.5	83.8	79.7	84.9	84.6	52.5	56.5	97.5
6.00	GEP(0.5,−1.9,0)	79.5	80.6	83.3	88.0	65.0	89.6	85.2	81.3	86.1	88.7	67.4	72.8	95.5
6.00	GEP(0.6,−1.3,0)	82.6	82.8	86.2	90.8	82.6	92.6	87.5	84.0	88.3	92.4	83.5	87.9	92.5
6.00	GEP(0.8,−0.8,0)	92.7	87.5	90.3	94.2	96.7	96.1	93.2	90.9	93.7	96.4	96.9	98.2	95.1
6.00	GEP(1,0,0)	98.4	92.6	93.7	96.7	99.6	98.3	97.7	96.6	97.8	98.5	99.6	99.8	98.9
6.00	GEP(2,3.9,0)	99.6	96.1	96.2	98.3	99.9	99.3	99.2	98.7	99.3	99.4	99.9	100	99.7
6.00	GEP(3,6.2,0)	99.8	97.3	97.3	98.9	99.9	99.6	99.5	99.2	99.5	99.6	100	100	99.8
6.00	GEP(0,140,129.1)	98.6	93.3	94.3	97.0	99.6	98.5	97.9	96.9	98.0	98.7	99.7	99.8	99.0
10.0	GEP(0.4,−1.7,0)	98.7	97.7	98.2	99.1	98.8	99.4	98.9	98.4	99.0	99.5	98.9	99.4	99.5
10.0	GEP(0.5,−1,0)	99.7	98.9	99.1	99.6	99.9	99.8	99.7	99.5	99.7	99.9	99.9	99.9	99.8
10.0	GEP(0.7,0,0)	100	99.9	99.8	99.9	100	100	100	100	100	100	100	100	100
10.0	GEP(1.4,5.7,0)	100	99.5	99.6	99.8	100	100	99.9	99.9	100	100	100	100	100
10.0	GEP(0,6,12.2)	99.9	99.1	99.1	99.6	100	99.8	99.8	99.7	99.8	99.9	100	100	99.9
10.0	GEP(0,17,−12.6)	99.9	99.1	99.2	99.7	100	99.9	99.8	99.7	99.8	99.9	100	100	99.9
Mean	$n = 200$	81.1	73.6	61.6	64.0	79.5	74.3	79.8	79.6	79.5	72.3	78.8	43.2	84.9
Rank		2	9	12	11	5.5	8	3	4	5.5	10	7	13	1
Dev. to the Best – Mean		8.0	15.5	27.4	25.1	9.5	14.8	9.3	9.5	9.5	16.7	10.3	45.8	4.2
Rank		2	9	12	11	5	8	3	5	5	10	7	13	1
Dev. to the Best – Max		49.7	67.1	100	84.5	77.3	75.8	44.3	45.6	43.8	86.9	78.8	100	39.4
Rank		5	6	12.5	10	8	7	3	4	2	11	9	12.5	1

Table 5
Power against symmetric distribution for 5% level tests, $M = 1,000,000$, $n = 20$, and $n = 50$

β_2	Alternative	$AD*$	Z_C	K^2	JB	T_w	$T_{K,5}$	W	CS	$BCMR$	β_3^2	R_{sJ}	S	R_n
3.00	N(0;1)	5.0	5.1	5.0	4.9	5.1	5.0	5.0	5.0	5.0	5.1	5.0	5.1	5.1
1.50	Beta(0.5;0.5)	61.5	77.7	48.0	0.6	54.8	53.4	72.4	75.2	67.0	77.8	46.0	89.1	50.4
1.51	LoConN(0.5;5)	81.3	72.0	55.2	1.1	79.4	66.8	75.3	78.2	71.3	58.6	52.8	65.3	68.0
1.63	JSB(0;0.5)	36.6	48.8	29.6	0.4	40.2	39.9	44.0	47.5	38.2	55.4	34.7	67.8	34.3
1.72	LoConN(0.5;4)	46.0	37.9	27.8	0.7	56.1	42.5	40.3	43.5	35.9	35.7	41.1	39.9	41.1
1.75	Tukey(1.5)	20.9	28.2	16.5	0.3	27.4	26.5	25.0	27.7	20.7	36.4	24.5	47.1	22.0
1.8	Tukey(2)	16.9	22.3	13.1	0.3	23.6	22.5	19.8	22.2	16.2	30.0	21.4	39.6	18.6
1.87	JSB(0;0.7)	13.2	15.8	9.7	0.3	20.3	18.5	14.5	16.4	11.7	22.4	18.5	29.4	15.4
2.04	LoConN(0.5;3)	15.0	12.3	8.2	0.7	23.4	16.3	13.0	14.5	10.9	14.4	19.9	17.1	14.9
2.06	Tukey(3)	6.1	6.9	3.7	0.3	9.7	8.3	6.4	7.3	5.0	10.7	9.4	15.8	8.0
2.10	Tukey(0.5)	6.6	6.4	3.8	0.4	11.0	8.7	6.3	7.1	5.1	9.5	10.4	13.0	8.1
2.14	Beta(2;2)	5.7	5.3	3.0	0.4	9.4	7.1	5.3	6.0	4.3	7.7	9.0	10.7	7.1
2.50	LoConN(0.5;2)	5.0	4.4	3.0	1.8	6.9	5	4.6	4.9	4.0	4.8	6.7	6.1	5.3
2.90	Tukey(5)	11.3	4.6	3.3	3.7	9.8	4.5	6.9	6.4	7.2	8.5	9.5	11.1	13.8
3.00	Tukey(0.1)	4.9	4.8	4.8	4.7	4.9	4.7	4.8	4.8	4.8	4.9	4.9	4.9	4.9
3.40	JSU(0;1;0;0.3)	6.5	7.3	8.2	8.4	6.4	7.6	7.1	6.9	7.3	7.2	6.4	6.5	6.6
3.91	GEP(1;-1.1;0)	9.0	10.6	12.1	12.5	8.4	11.1	10.1	9.8	10.6	10.1	8.5	8.4	8.9
4.00	Student-t(10)	8.8	10.2	11.8	12.2	8.6	10.9	9.8	9.5	10.3	10.1	8.6	8.8	9.1
4.20	Logistic(0;1)	10.5	12.0	14.1	14.7	10.3	13.0	11.6	11.2	12.3	12.3	10.4	10.6	10.6
4.51	JSU(0;1;0;0.5)	11.2	12.9	15.0	15.6	10.9	13.9	12.4	12.0	13.1	12.9	11.1	11.2	11.4
5.40	Tukey(10)	91.0	64.0	48.4	53.1	74.3	59.6	80.7	78.5	82.2	74.6	89.9	75.2	83.3
6.00	Laplace(0;1)	27.2	24.7	28.5	30.1	27.4	28.3	26.1	25.0	27.8	30.0	28.7	28.2	25.8
10.4	Tukey(20)	100	98.3	89.8	92.3	93.1	93.8	99.7	99.7	99.8	96.7	99.9	93.8	99.2
36.19	JSU(0;1;0;1)	42.3	41.2	45.0	46.7	41.7	44.9	42.4	41.3	44.1	44.6	43.4	42.4	40.4
75.98	JSU(0;1;0;1.1)	50.0	47.6	51.1	53.0	49.1	51.5	49.4	48.2	51.1	51.7	51.3	49.8	47.5
Inf	Student-t(4)	22.4	24.4	27.6	28.7	22.1	26.5	24.1	23.4	25.3	25.0	22.6	22.6	22.2
Inf	Student-t(2)	52.7	51.6	54.9	56.5	51.8	54.9	52.8	51.8	54.4	54.3	53.7	52.4	50.7
Inf	Student-t(1)	88.1	84.4	84.7	86.1	86.0	85.9	86.6	86.0	87.4	86.7	88.7	86.3	85.5
Mean	$n = 20$	30.6	30.1	25.9	18.9	31.1	29.7	30.6	31.1	29.8	32.1	29.9	34.2	29.2
Rank		5.5	7	12	13	3.5	10	5.5	3.5	9	2	8	1	11
Dev. to the Best – Mean		7.4	7.9	12.0	19.0	6.8	8.2	7.4	6.9	8.2	5.9	8.0	3.7	8.7
Rank		6	7	12	13	3	9.5	5	4	9.5	2	8	1	11
Dev. to the Best – Max		31.2	27.0	42.6	88.5	34.3	35.7	23.8	20.3	29.6	22.7	43.1	16.2	38.7
Rank		7	5	11	13	8	9	4	2	6	3	12	1	10
3.00	N(0;1)	5.0	5.0	5.1	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0
1.50	Beta(0.5;0.5)	99.1	100	99.5	38.3	96.4	96.5	99.9	100	99.9	100	93.2	100	95.3
1.51	LoConN(0.5;5)	100	99.6	99.4	43.3	99.5	97.5	99.9	99.9	99.8	98.4	90.1	89.2	98.5
1.63	JSB(0;0.5)	90.1	98.7	96.3	9.4	88.0	90.3	97.2	98.2	95.7	99.3	84.6	99.5	83.5
1.72	LoConN(0.5;4)	94.2	86.0	89.8	5.6	96.5	86.5	90.1	92.4	88.2	88.0	86.9	62.3	90.2
1.75	Tukey(1.5)	67.8	89.9	84.9	1.5	71.5	74.7	84.0	88.3	79.0	94.8	68.8	95.1	62.1
1.8	Tukey(2)	57.7	82.6	78.0	0.8	64.6	66.8	75.0	80.7	68.8	90.7	62.3	90.3	53.6
1.87	JSB(0;0.7)	45.2	66.1	66.8	0.3	56.8	56.0	58.4	65.5	51.7	80.7	54.8	74.8	43.7
2.04	LoConN(0.5;3)	44.4	34.4	43.7	0.3	62.2	39.4	37.7	43.1	33.7	47.0	57.0	24.5	41.9
2.06	Tukey(3)	16.6	33.8	31.3	0.0	25.0	22.9	26.4	32.6	20.9	49.3	24.8	47.7	17.2
2.10	Tukey(0.5)	17.3	24.3	29.2	0.0	29.3	22.6	20.9	26.1	16.8	41.0	28.6	29.9	17.8
2.14	Beta(2;2)	13.3	17.5	21.7	0.1	23.7	16.6	15.3	19.4	12.1	31.5	23.3	21.5	13.8
2.50	LoConN(0.5;2)	7.8	6.1	7.2	0.8	12.7	5.6	6.5	8.0	5.5	9.6	12.4	5.0	6.8
2.90	Tukey(5)	24.8	4.1	0.8	1.1	19.8	2.9	13.0	11.7	11.7	5.3	20.2	28.0	33.3
3.00	Tukey(0.1)	4.9	4.3	4.3	4.2	4.8	4.3	4.6	4.6	4.5	4.7	4.8	4.8	4.5
3.40	JSU(0;1;0;0.3)	7.5	9.6	10.6	12.0	8.0	11.7	9.2	8.5	9.7	9.7	8.1	10.0	9.6
3.91	GEP(1;-1.1;0)	12.2	16.5	18.4	21.1	12.4	20.9	15.9	14.5	16.9	17.4	12.7	15.6	16.8

(*continued*)

Table 5
Continued

β_2	Alternative	$AD*$	Z_C	K^2	JB	T_w	$T_{K,5}$	W	CS	$BCMR$	β_3^2	R_{sJ}	S	R_n
4.00	Student-t(10)	12.1	16.3	18.2	20.6	13.7	20.5	15.6	14.3	16.6	17.2	13.9	17.3	16.5
4.20	Logistic(0;1)	16.1	19.9	22.5	25.7	18.9	26.0	19.7	17.9	21.0	23.0	19.1	23.7	21.0
4.51	JSU(0;1;0;0.5)	17.4	22.1	24.7	28.0	20.3	28.3	21.7	19.9	23.0	24.8	20.5	25.1	23.0
5.40	Tukey(10)	100	94.9	66.3	78.8	98.6	92.8	99.7	99.5	99.7	96.9	100	99.0	99.7
6.00	Laplace(0;1)	54.6	45.5	48.9	55.4	62.7	59.3	52.1	48.3	54.2	60.8	64.0	69.3	57.5
10.4	Tukey(20)	100	100	99.6	99.9	100	100	100	100	100	100	100	100	100
36.19	JSU(0;1;0;1)	76.0	72.6	74.9	79.4	80.3	81.6	76.4	73.9	77.8	81.4	81.1	84.1	78.4
75.98	JSU(0;1;0;1.1)	84.3	80.3	81.8	85.6	87.6	87.6	84.0	82.0	85.0	87.8	88.3	90.2	85.6
Inf	Student-t(4)	42.0	46.1	49.5	53.9	47.2	55.1	46.9	44.3	48.7	52.1	47.7	52.9	48.8
Inf	Student-t(2)	85.9	84.0	85.0	88.2	88.6	89.6	86.4	84.8	87.3	89.3	89.1	90.8	87.5
Inf	Student-t(1)	99.7	99.3	99.2	99.5	99.8	99.7	99.6	99.6	99.7	99.7	99.8	99.9	99.7
Mean	$n = 50$	49.9	52.1	52.1	30.7	53.4	52.2	52.2	53.0	51.2	57.3	52.2	55.6	50.4
Rank		12	8.5	8.5	13	3	6	6	4	10	1	6	2	11
Dev. to the Best – Mean		10.9	8.6	8.6	30.0	7.4	8.5	8.5	7.8	9.5	3.4	8.5	5.2	10.3
Rank		12	8.5	8.5	13	3	6	6	4	10	1	6	2	11
Dev. to the Best – Max		35.5	29.2	33.7	93.6	26.1	30.4	24.5	21.6	29.0	28.0	28.4	37.7	37.1
Rank		10	7	9	13	3	8	2	1	6	4	5	12	11

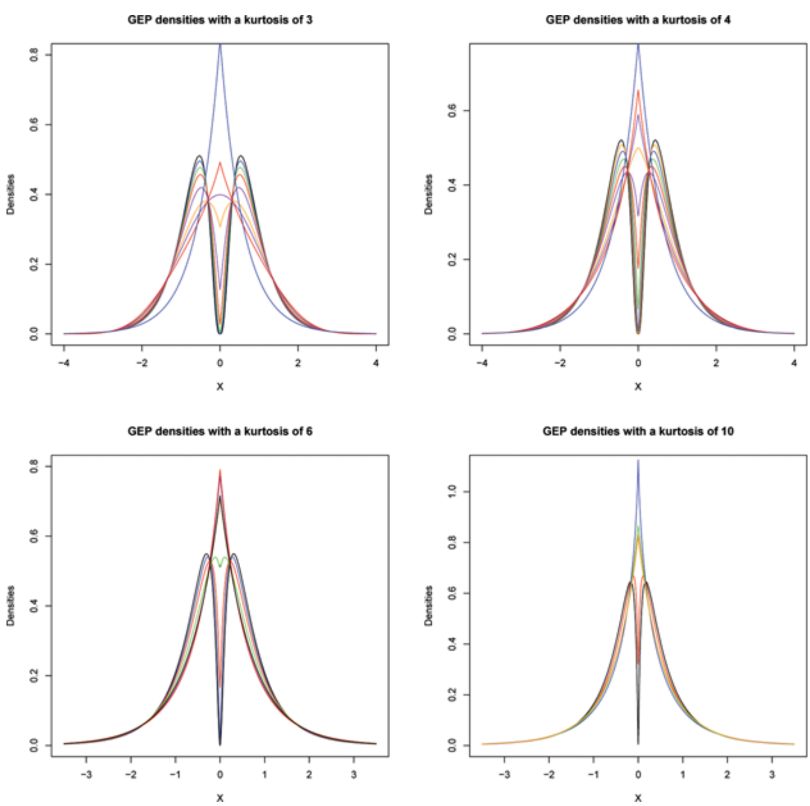

Figure 2. GEP densities with different tails behavior. (color figure available online.)

Table 6
Power against symmetric distribution for 5% level tests, $M = 1,000,000$, $n = 100$, and $n = 200$

β_2	Alternative	$AD*$	Z_C	K^2	JB	T_w	$T_{K,5}$	W	CS	$BCMR$	β_3^2	R_{sJ}	S	R_n
3.00	N(0;1)	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.1	5.0	5.0	5.0	5.0	5.0
1.50	Beta(0.5;0.5)	100	100	96.8	100	100	100	100	100	100	100	99.9	100	100
1.51	LoConN(0.5;5)	100	100	97.3	100	100	100	100	100	100	100	99.3	93.0	100
1.63	JSB(0;0.5)	100	100	100	98.8	99.7	99.9	100	100	100	100	99.5	100	99.7
1.72	LoConN(0.5;4)	100	99.6	99.9	91.8	100	99.7	99.9	99.9	99.9	99.7	99.2	64.1	99.9
1.75	Tukey(1.5)	98.0	100	99.9	85.4	96.8	99.0	99.9	100	99.8	100	96.3	100	95.9
1.8	Tukey(2)	94.9	99.9	99.7	74.2	94.2	97.8	99.6	99.8	99.3	100	93.6	99.7	91.9
1.87	JSB(0;0.7)	87.0	99.0	98.6	55.0	90.0	94.7	97.1	98.5	95.7	99.7	89.2	95.5	84.5
2.04	LoConN(0.5;3)	81.8	69.6	82.4	22.4	92.7	77.2	75.0	80.1	72.4	82.8	90.4	18.9	80.1
2.06	Tukey(3)	44.2	88.4	79.6	10.5	51.6	61.8	76.6	84.4	70.0	92.6	51.6	78.0	42.2
2.10	Tukey(0.5)	42.6	71.1	75.8	8.8	59.0	60.1	60.2	69.6	53.9	86.0	58.6	42.2	42.9
2.14	Beta(2;2)	31.8	55.8	63.1	4.5	49.0	46.7	45.4	55.3	39.3	75.2	48.7	26.5	32.2
2.50	LoConN(0.5;2)	13.4	10.9	15.7	0.8	23.7	10.7	11.2	14.6	9.7	19.3	23.5	1.7	11.7
2.90	Tukey(5)	52.0	16.5	0.3	0.4	38.7	2.6	37.8	39.3	33.0	3.1	39.1	52.7	68.1
3.00	Tukey(0.1)	4.9	3.9	3.9	3.8	4.7	3.8	4.3	4.4	4.2	4.4	4.7	4.8	4.2
3.40	JSU(0;1;0;0.3)	8.5	12.4	13.4	16.2	10.2	16.3	11.7	10.4	12.5	13.3	10.2	14.8	13.1
3.91	GEP(1;-1.1;0)	16.5	23.8	26.3	31.5	18.1	32.5	23.4	20.6	25.1	28.1	18.3	25.0	27.1
4.00	Student-t(10)	16.2	23.7	25.9	30.7	21.0	31.5	22.9	20.4	24.4	27.2	21.0	28.7	25.7
4.20	Logistic(0;1)	23.9	29.7	33.2	39.4	31.4	41.1	30.5	27.0	32.4	37.9	31.4	40.9	34.1
4.51	JSU(0;1;0;0.5)	26.2	33.6	36.9	43.0	33.8	44.7	34.0	30.5	35.9	41.0	33.8	43.1	37.6
5.40	Tukey(10)	100	100	89.2	96.4	100	99.8	100	100	100	99.9	100	100	100
6.00	Laplace(0;1)	82.7	69.7	72.1	79.9	90.2	84.7	79.7	76.0	81.0	86.2	90.6	94.0	85.4
10.4	Tukey(20)	100	100	100	100	100	100	100	100	100	100	100	100	100
36.19	JSU(0;1;0;1)	95.3	93.0	93.8	96.1	97.3	97.2	95.5	94.4	95.8	97.3	97.4	98.4	96.5
75.98	JSU(0;1;0;1.1)	98.2	96.6	96.9	98.2	99.1	98.8	98.1	97.6	98.3	98.9	99.1	99.5	98.6
Inf	Student-t(4)	65.1	68.8	71.9	77.4	73.4	79.4	71.2	67.9	72.8	78.0	73.5	79.8	74.4
Inf	Student-t(2)	98.4	97.7	98.0	98.8	99.1	99.1	98.5	98.2	98.7	99.2	99.1	99.5	98.9
Inf	Student-t(1)	100	100	100	100	100	100	100	100	100	100	100	100	100
Mean	$n = 100$	63.8	66.7	67.0	56.0	67.1	67.3	67.1	67.6	66.4	70.5	66.9	64.5	66.1
Rank		12	8	6	13	4.5	3	4.5	2	9	1	7	11	10
Dev. to the Best – Mean		10.7	7.7	7.5	18.4	7.4	7.2	7.4	6.8	8.1	4.0	7.6	10.0	8.4
Rank		12	8	6	13	4.5	3	4.5	2	9	1	7	11	10
Dev. to the Best – Max		48.4	51.6	67.8	82.1	41.0	65.5	30.3	28.8	35.9	65.0	41.0	73.8	50.4
Rank		6	8	11	13	4.5	10	2	1	3	9	4.5	12	7
3.00	N(0;1)	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0
1.50	Beta(0.5;0.5)	100	100	49.2	100	100	100	100	100	100	100	100	100	100
1.51	LoConN(0.5;5)	100	100	54.7	100	100	100	100	100	100	100	100	92.4	100
1.63	JSB(0;0.5)	100	100	97.7	100	100	100	100	100	100	100	100	100	100
1.72	LoConN(0.5;4)	100	100	99.8	100	100	100	100	100	100	100	100	55.1	100
1.75	Tukey(1.5)	100	100	100	100	100	100	100	100	100	100	100	100	100
1.8	Tukey(2)	100	100	100	100	99.9	100	100	100	100	100	99.9	100	100
1.87	JSB(0;0.7)	99.9	100	100	99.9	99.7	100	100	100	100	100	99.7	99.8	99.9
2.04	LoConN(0.5;3)	99.3	96.3	98.9	89.3	99.9	98.8	98.4	98.9	98.1	99.0	99.8	8.1	99.2
2.06	Tukey(3)	88.8	100	99.6	88.4	85.0	97.7	99.9	100	99.7	100	84.7	96.4	89.0
2.10	Tukey(0.5)	84.4	99.5	99.2	84.0	90.3	97.3	97.9	99.1	96.8	99.9	90.0	51.4	86.7
2.14	Beta(2;2)	71.1	97.2	96.9	68.0	82.5	92.4	92.4	9.06	89.5	99.2	82.1	26.7	74.6
2.50	LoConN(0.5;2)	27.1	22.0	33.7	7.3	44.8	27.3	23.5	29.1	21.3	39.2	44.2	0.1	25.1
2.90	Tukey(5)	88.5	75.6	0.3	0.1	69.9	3.3	88.5	91.2	84.5	2.6	70.5	80.7	96.9
3.00	Tukey(0.1)	4.9	3.5	3.7	3.4	4.6	3.3	4.1	4.2	4.0	4.2	4.6	4.8	3.8
3.40	JSU(0;1;0;0.3)	10.5	16.3	18.4	22.4	14.2	23.5	15.9	13.3	17.1	19.5	14.3	21.0	18.4
3.91	GEP(1;-1.1;0)	25.6	35.4	40.3	47.4	28.4	50.3	37.2	32.0	39.2	46.3	28.8	37.7	44.3

(continued)

Table 6
Continued

β_2	Alternative	$AD*$	Z_C	K^2	JB	T_w	$T_{K,5}$	W	CS	$BCMR$	β_3^2	R_{sJ}	S	R_n
4.00	Student-t(10)	24.5	35.1	39.5	45.9	34.3	48.4	35.8	31.2	37.7	44.1	34.4	44.1	40.3
4.20	Logistic(0;1)	39.4	45.1	51.4	59.0	52.5	62.7	49.2	43.5	51.1	60.6	52.7	62.8	54.4
4.51	JSU(0;1;0;0.5)	42.7	50.9	56.7	63.8	55.8	67.1	54.1	48.6	56.0	64.7	55.9	65.5	59.1
5.40	Tukey(10)	100	100	99.7	100	100	100	100	100	100	100	100	100	100
6.00	Laplace(0;1)	98.4	92.5	93.7	96.6	99.6	98.3	97.5	96.5	97.7	98.5	99.6	99.8	98.9
10.4	Tukey(20)	100	100	100	100	100	100	100	100	100	100	100	100	100
36.19	JSU(0;1;0;1)	99.9	99.7	99.8	99.9	100	99.9	99.9	99.8	99.9	100	100	100	99.9
75.98	JSU(0;1;0;1.1)	100	99.9	99.9	100	100	100	100	100	100	100	100	100	100
Inf	Student-t(4)	89.2	90.0	92.4	94.8	94.0	96.0	92.5	90.6	93.1	95.8	94.1	96.2	94.1
Inf	Student-t(2)	100	100	100	100	100	100	100	100	100	100	100	100	100
Inf	Student-t(1)	100	100	100	100	100	100	100	100	100	100	100	100	100
Mean	$n = 200$	75.0	77.3	72.5	74.1	77.2	77.5	78.3	77.8	78.2	77.8	77.2	69.6	78.2
Rank		10	7	12	11	8.5	6	1	4.5	2.5	4.5	8.5	13	2.5
Dev. to the Best – Mean		7.1	4.8	9.6	8.0	4.9	4.6	3.8	4.3	3.9	4.3	4.9	12.5	3.9
Rank		10	7	12	11	8.5	6	1	4.5	2.5	4.5	8.5	13	2.5
Dev. to the Best – Max		28.1	22.8	96.6	96.8	27.0	93.6	21.3	19.3	23.5	94.3	26.4	91.8	24.6
Rank		8	3	12	13	7	10	2	1	4	11	6	9	5

Note that we have included, in these tables, results for the GEP$(1, -1.1, 0)$ alternative. Our reason for doing so is that the Student-$t(10)$ density is virtually undistinguishable from that GEP density. It is interesting to note that all tests perform the same for both of these alternatives, that is, for the Student density and its approximation by a GEP density. Again, this suggests that a test of normality against GEP alternatives should fare well against non GEP alternatives that can be approximated by a member of the GEP family of distributions.

In Tables 3–6, we report the empirical power (as measured by the proportion of simulated samples for which the composite hypothesis of normality was rejected) under the considered alternatives. For our simulation study, we generated 1,000,000 samples under each alternative (except for the GEP alternatives for which we generated 100,000 samples) and for sample sizes of $n = 20$, 50, 100, and 200 at a significance level of 5%. We also report three measures of performance for each test in each simulation. First, we report the average power against all alternatives. Then, for each alternative, we also consider for each test the difference of its power with the best one among all the tests for this specific alternative. These differences are labeled in the tables as "Deviation to the Best" and, for each test, we report its mean and maximum. The highest the first measure, and the smallest the last two measures, the best is the performance of a test. We also report the rank of each test among the 13 tests, for each one of the 3 quality measures.

When GEP alternatives are considered, our test R_n has the best rank for the 3 measures for $n = 100$ and $n = 200$. For $n = 20$ and $n = 50$, its rank alternates between 2nd and 3rd for the three measures; see Tables 3 and 4. These results clearly suggest that the test based on R_n has the best performance among the 13 tests against the GEP alternatives $n \geq 100$ and one of the best performances for $n \leq 50$.

When the 28 usual alternatives are considered, our test R_n performs relatively well, see Tables 5 and 6. Is is more difficult to make some conclusions with this more limited set of alternatives, but it seems that our test becomes better as n increases, ranking among the bests for $n = 200$.

In summary, our simulations suggest that our test, specifically designed to detect non normality in the tails for symmetric densities, has the best results for symmetric alternatives. The performance seems to improve with the sample size, but the dominance of the test is already visible for small sample sizes.

5. Conclusion

We proposed a new test of normality based on using Rao's score test on a GEP family of distributions which includes the normal as a special case. This test is tailored to detect departures from normality in the tails of the distribution. Our goal was to provide a simple and intuitive test of normality against symmetric alternatives with non normal tails. We have obtained a test which is the most powerful against GEP alternatives compared to the best normality tests previously available.

Appendix: Proofs

We first state a lemma needed for the proof of Proposition 3.1.

Lemma A.1. *If $X \sim N(\mu, \sigma^2)$, the vector $\boldsymbol{r}_n(\mu, \sigma)$ defined by* (4) *satisfies*

$$\frac{\partial}{\partial \mu}\boldsymbol{r}_n(\mu, \sigma) = o_P(1)\mathbf{1}_3 \quad \text{and} \quad \frac{\partial}{\partial \sigma}\boldsymbol{r}_n(\mu, \sigma) = \sigma^{-1}\boldsymbol{v}_0 + o_P(1)\mathbf{1}_3.$$

Proof. In order to prove this result, we work from the fact that

$$\boldsymbol{r}_n(\mu, \sigma) = \frac{1}{n}\sum_{i=1}^{n}\boldsymbol{d}_0(Y_i).$$

Note that we can write $\boldsymbol{d}_0(y) = (d_1(y), d_2(y), d_3(y))^T$ and that the following properties hold:

(i) $d_j(y)$ is an even function;
(ii) $d_j(Y)$ is almost everywhere differentiable, implying that $\boldsymbol{d}_0(Y)$ and $\boldsymbol{r}_n(\mu, \sigma)$ are differentiable with probability 1;
(iii) $\mathbb{E}_0\left[d_j(Y)\right] = \mathbb{E}_0\left[Yd_j(Y)\right] = 0$;
(iv) $\mathbb{E}_0\left[Y^2|d_j(Y)|\right] < \infty$;
(v) $\mathbb{E}_0\left[|d'_j(Y)|\right] < \infty$; and
(vi) $\mathbb{E}_0\left[|Yd'_j(Y)|\right] < \infty$,

where $d'_j(y) = \frac{d}{dy}d_j(y)$, for $j = 1, 2, 3$. On the other hand, by the weak law of large numbers, we have

$$\begin{aligned}\frac{\partial}{\partial \mu}\boldsymbol{r}_n(\mu, \sigma) &= \mathbb{E}_0\left[\frac{\partial}{\partial \mu}\boldsymbol{d}_0(Y)\right] + o_P(1)\mathbf{1}_3 \\ &= \mathbb{E}_0\left[\frac{\partial}{\partial \mu}\left(\boldsymbol{d}_0\left(\frac{X-\mu}{\sigma}\right)\right)\right] + o_P(1)\mathbf{1}_3,\end{aligned}$$

a similar result holding for the partial derivative with respect to σ. Note that properties (v) and (vi) are used here to ensure the weak law of large numbers can be applied. In order to calculate the last expectation, first note that

$$\mathbb{E}_0\left[\frac{\partial}{\partial\mu}d_j\left(\frac{X-\mu}{\sigma}\right)\right] = -\sigma^{-1}\mathbb{E}_0\left[d_j'(Y)\right].$$

Now, recall Stein's famous Lemma which, in the case of the standard normal distribution, reduces to

$$\mathbb{E}_0\left[h'(Y)\right] = \mathbb{E}_0\left[Yh(Y)\right],$$

whenever these expectations exist (cf. Stein, 1981). Using this result with properties (iii) and (v) leads to

$$\mathbb{E}_0\left[d_j'(Y)\right] = \mathbb{E}_0\left[Yd_j(Y)\right] = 0,$$

for $j = 1, 2, 3$ so that the first result is proved. In the same way, it is easy to verify that

$$\mathbb{E}_0\left[\frac{\partial}{\partial\sigma}d_j\left(\frac{X-\mu}{\sigma}\right)\right] = -\sigma^{-1}\mathbb{E}_0\left[Yd_j'(Y)\right].$$

Using Stein's Lemma one more time with $h(Y) = Yd_j(Y)$ and $h'(Y) = d_j(Y) + Yd_j'(Y)$, and using properties (iii), (iv), and (vi), we have

$$\mathbb{E}_0\left[Yd_j'(Y)\right] = \mathbb{E}_0\left[d_j(Y) + Yd_j'(Y)\right] = \mathbb{E}_0\left[Y^2 d_j(Y)\right],$$

for $j = 1, 2, 3$ so that the second result also holds.

Proof of Proposition 3.1. Under normality, it is well known that

$$\overline{X}_n - \mu = O_P(n^{-1/2}). \tag{10}$$

As a result, it is easy to verify that

$$S_n^2 - \sigma^2 = -\sigma^2\left(1 - \frac{1}{n}\sum_{i=1}^{n} Y_i^2\right) - (\overline{X}_n - \mu)^2 = -\sigma^2\left(1 - T_n\right) + o_P(n^{-1/2}),$$

so that, upon applying the δ-method, we get

$$S_n - \sigma = \frac{1}{2\sigma}(S_n^2 - \sigma^2) + o_P(n^{-1/2}) = -\frac{\sigma}{2}\left(1 - T_n\right) + o_P(n^{-1/2}). \tag{11}$$

Note that $1 - T_n = O_P(n^{-1/2})$ by the CLT. Now, to obtain the wanted result, we use a Taylor expansion in probability of $\boldsymbol{r}_n(\overline{X}_n, S_n)$ along with (10), (11), and Lemma A.1. This leads to

$$\begin{aligned}\boldsymbol{r}_n(\overline{X}_n, S_n) &= \boldsymbol{r}_n(\mu, \sigma) + (\overline{X}_n - \mu)\frac{\partial}{\partial\mu}\boldsymbol{r}_n(\mu, \sigma) + (S_n - \sigma)\frac{\partial}{\partial\sigma}\boldsymbol{r}_n(\mu, \sigma) + o_P(n^{-\frac{1}{2}})\mathbf{1}_3 \\ &= \boldsymbol{r}_n(\mu, \sigma) - \frac{1}{2}\left(1 - T_n\right)\boldsymbol{v}_0 + o_P(n^{-\frac{1}{2}})\mathbf{1}_3.\end{aligned}$$

To conclude, it is sufficient to evaluate the vector $\boldsymbol{v}_0$ numerically, where $\boldsymbol{v}_0$ is given in Eq. (7).

Proof of Theorem 3.1. First, note that a proof of this theorem could also be obtained using general results on score tests exposed in Cox and Hinkley (1979).

Now, note that Proposition 3.1 allows us to write

$$n^{1/2}\boldsymbol{r}_n(\overline{X}_n, S_n) = n^{1/2}\boldsymbol{M}\bar{\boldsymbol{Z}}_n + o_P(1)\mathbf{1}_3, \tag{12}$$

where $\bar{\boldsymbol{Z}}_n = \frac{1}{n}\sum_{i=1}^n \boldsymbol{Z}_i$, and where the matrix $\boldsymbol{M}$ and random vectors $\boldsymbol{Z}_i$ are defined as

$$\boldsymbol{M} = \left(I_3; -\frac{1}{2}\boldsymbol{v}_0\right) \quad \text{and} \quad \boldsymbol{Z}_i = \begin{pmatrix} \boldsymbol{d}_0(Y_i) \\ 1 - Y_i^2 \end{pmatrix},$$

and where $\boldsymbol{d}_0(y)$ and $\boldsymbol{v}_0$ are defined, respectively, in (3) and (7).

On the other hand, under the null hypothesis, the central limit theorem implies that

$$n^{1/2}\bar{\boldsymbol{Z}}_n \xrightarrow{\mathscr{D}} N_4\left(\mathbf{0}, \begin{pmatrix} \boldsymbol{J}_0 & \boldsymbol{v}_0 \\ \boldsymbol{v}_0^T & 2 \end{pmatrix}\right), \tag{13}$$

since

$$\mathbb{E}_0\left[(1 - Y^2)\boldsymbol{d}_0(Y)\right] = -\mathbb{E}_0\left[Y^2\boldsymbol{d}_0(Y)\right] = \boldsymbol{v}_0.$$

Finally, (12) and (13) together imply that

$$n^{1/2}\boldsymbol{r}_n(\overline{X}_n, S_n) \xrightarrow{\mathscr{D}} N_3\left(\mathbf{0}, \boldsymbol{J}_0 - \frac{1}{2}\boldsymbol{v}_0\boldsymbol{v}_0^T\right).$$

The convergence in distribution of R_n follows directly. □

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